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# Granular Flows

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**Abstract.** Those who have played with sand on the beach or with sugar in their kitchen are aware that a collection of solid grains can behave macroscopically like a liquid and flow. However, the description of this peculiar fluid still represents a challenge due to the lack of constitutive laws able to describe the rich phenomenology observed. In this paper we first review the properties of dry granular flows and we present recent advances in our understanding of their rheological behavior. The success and limits of a simple visco-plastic model recently developed is presented. In a second part of the paper, we go beyond the simple and ideal situation where the material is made of grains having the same size and interacting by contact interactions only. We present studies on more complex and realistic granular medium like polydispersed media, like cohesive granular media and like granular pastes made of grains immersed in a liquid. We analyse to which extend the progress made in our description of monodispersed dry granular flows can help understanding these more complex granular media.

## 1 Introduction

Sand, gravels, rice, sugar...Granular matter is everywhere in our every day life (Fig. 1). Strong enough to support buildings, a granular medium can flow like a liquid for example in an hourglass, or can be transported by the wind to create dunes in the desert. This variety of behaviour represents one of the difficulty of the physics of granular media [45]. Research in this area is motivated by numerous applications encountered in industrial processes and especially in geophysics for the description and prediction of natural hazards like landslides, rock avalanches or pyroclastic flows. However, the recent interest in granular flows is certainly also stimulated by new fundamental questions raised by this peculiar fluid, which shares similarities with other athermal disordered systems such as foam, amorphous solids or emulsions [63, 17] and which exhibits a very rich phenomenology [5].

A granular medium is a collection of macroscopic particles, their size being typically greater than 100 $\mu$m. This limitation in size corresponds to a limitation in the type of interaction between particles. A granular medium means non brownian particles, which interact solely by friction and collision. For smaller particles, other interactions like Van der Waals forces, like humidity start to play a role and one enters the world of powder. At even smaller sizes, below 1 $\mu$m, thermal agitation is no longer negligible and one enters the world of colloids. Granular media are then a priori simple systems made of solid particles interacting trough contact interactions. However, they still resists our understanding and no theoretical framework

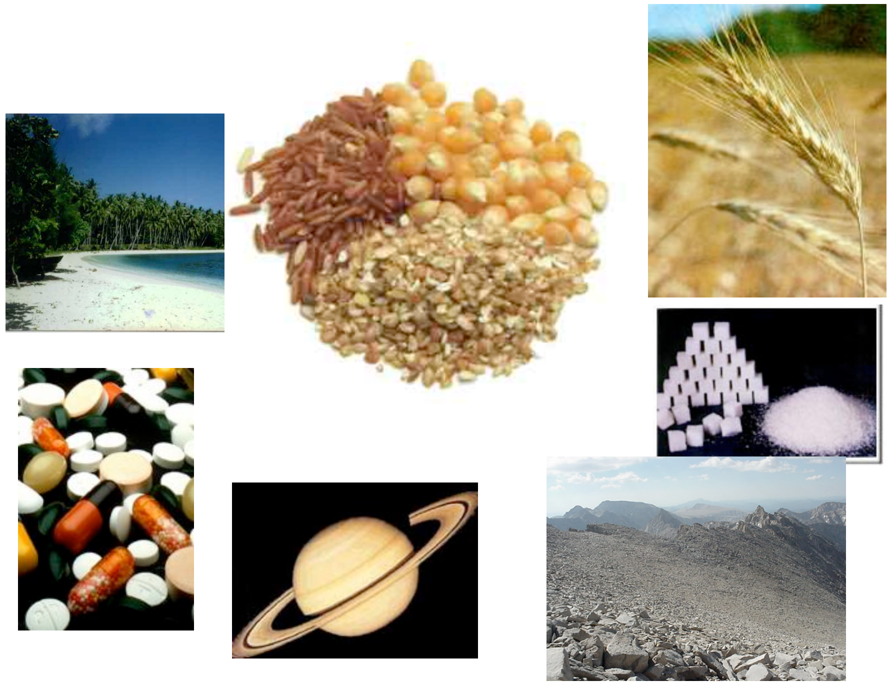


Figure 1: Examples of granular materials.

is available to describe the variety of behaviours observed. One can try to list the difficulties encountered when dealing with granular material.

First, granular media are composed of a large number of particles. A spoon of sugar contains more than one millions of grains, which is larger than what we can compute numerically with ideal spherical particles. There is then a need for a continuous description, trying to define averaged quantities and to model the granular medium as a continuum medium. A large number of particles is not necessary a serious obstacle, if one consider gases or liquids, for which the number of molecules is much larger than the number of grains in a spoon of sugar. However, in the case of gases or liquids, the presence of thermal agitation allows a proper statistical approach, allowing to derive macroscopic quantities from microscopic ones. In the case of a granular medium, particles are too large to experience Brownian motion and the statistical average over different configurations is not possible. The system is stuck in metastable states. Granular media are then often qualified as athermal systems [63]. Another difficulty encountered when one try to apply tools from statistical physics to granular media, is the dissipative nature of the particle interaction. Contact interactions including friction and inelastic collisions are highly non linear and dissipative mechanisms. This dissipation at the microscopic level is an important difference with classical systems studied in statistical physics. The continuum description of granular media is also made difficult by the lack of a clear scale separation between the microscopic scale i.e. the grains size, and the macroscopic scale. i.e. the size of the flow. Typically, when sand flows down on a pile, the flow thickness is about 10 to 20 particle diameters. The physics of granular media shares this difficulty with nanofluidics, when effects of the size of the molecule start to play a role. The last difficulty is the observation that granular media exists under different states [83]. Depending on the way it is handled, a granular material can behave like a solid, a liquid or a gas (Fig. 2). Grains can sustain stresses and create a static pile, but can also flow like a liquid in an hourglass, or can create a gas when they are strongly agitated. These different flow regimes can coexist in a single configuration as shown by the flow of beads on a pile shown in Fig. 2.

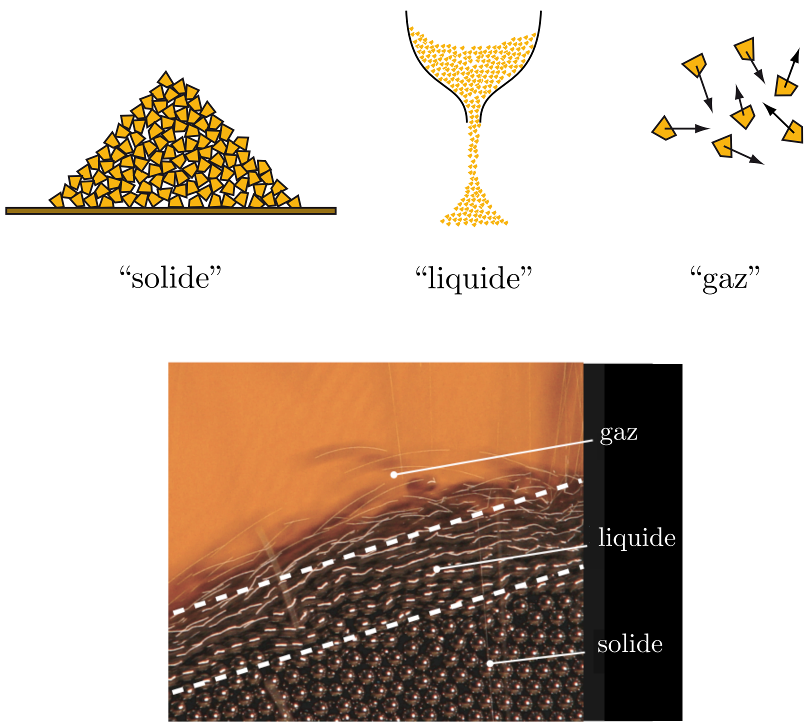



Figure 2: Different states of granular media.

In front of such a complexity, different frameworks have been developed to describe the different flow regimes. The dense quasi-static regime where the deformations are very slow and the particles interact by frictional contacts is usually described using plasticity theories [100, 109, 91]. The gaseous regime where the flow is very rapid and dilute and the particles interact by collision [38] has motivated a lot of work based on the concept of the kinetic theory of gases. The intermediate liquid regime where the material is dense but still flows like a liquid is the less understood regime [34]. In this paper we focus on this last liquid regime, which is most often encountered in applications, and discuss the possibility of a hydrodynamic description of dense granular flows.

## 2 The granular liquid

Rock avalanches, flows of cereals out of a silo are typical examples of dense granular flows. In this flow regime, the volume fraction of the material (ratio of the volume occupied by the grains to the total volume) is high and close to the maximum value.

The grains interact both by friction and collision through a contact network. From a phenomenological point of view, the material flows like a liquid with peculiar features. To better understand this regime, different flow configurations have been investigated, the most common being sketched in Figure 3. They can be divided in two families : flows confined between walls as in shear cells or silo, and free surface flows like flows down inclined plane, flows in rotating drums or flows on a pile. Their characteristics in terms of velocity profiles, density profiles, velocity fluctuations are discussed in details in [36]. Recently, by analogy with classical hydrodynamics problems, more complex flow configurations have been analyzed like dam break problems [59, 68], coating-like problems [28, 22], mixing experiments [76], split Couette devices [30], drag problems [55] and instabilities [5].

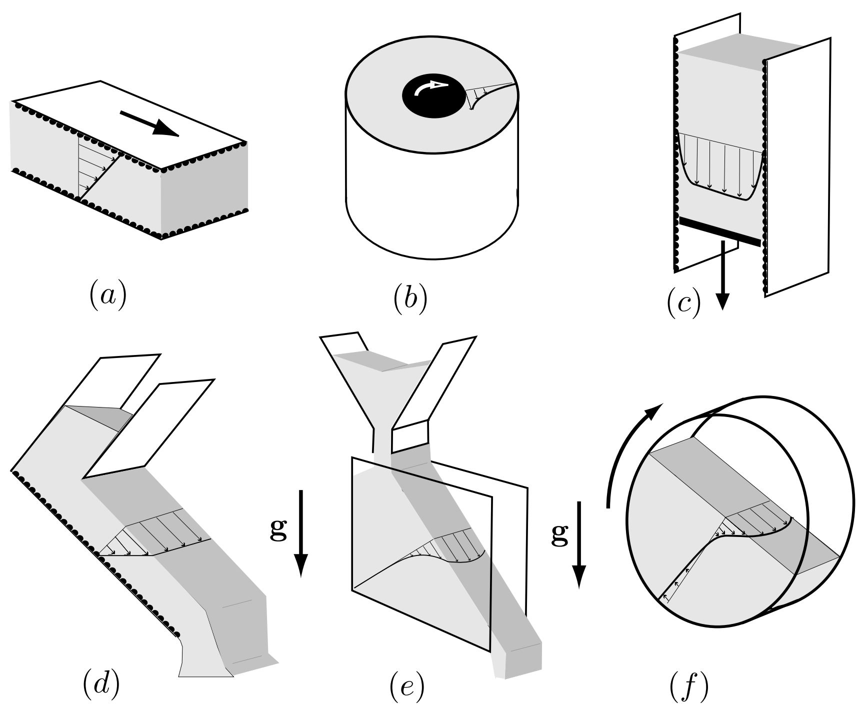



Figure 3: Different configurations used to study granular flows. (a) Plane shear (b) Couette cell (c) vertical silo (d) Inclined plane (e) Heap flow (f) rotating drum.

A recurrent and central question underlying all the studies is the question of the constitutive equations of this peculiar liquid. Dense granular flows belong to the visco-plastic family of materials because of two broad properties. First, a flow threshold exists, although it is expressed in term of friction instead of a yield stress, as in a classical visco-plastic material. Second, when the material is flowing, shear rate dependence is observed, which gives a viscous-like behavior. In the following section we present recent advances in our understanding of the rheology of dense granular flows. We first discuss the plane shear configuration, which provides the basic ideas allowing to propose a constitutive law for dense granular flows. The application to other configurations is discussed and the limits of this simple local rheology is discussed.

### 2.1 Local rheology

Let us consider a granular material made of particles of diameter $d$ and density $\rho_p$ under a confinement pressure $P$. The material is confined between two rough plates by a pressure $P$ imposed to the top plate. The material is sheared at a given shear rate $\dot{\gamma} = V_w/L$ imposed by the relative displacement of the top plate at a velocity $V_w$. (Fig. 4). In absence of gravity, the force balance implies that both the shear stress $\tau = \sigma_{xz}$ and the normal stress $P = \sigma_{zz}$ are homogeneous across the cell. This configuration is then the simplest configuration to study the rheology of granular flows, namely to study how the shear stress $\tau$ and the volume fraction $\phi$ vary with the shear rate $\dot{\gamma}$ and the pressure $P$.

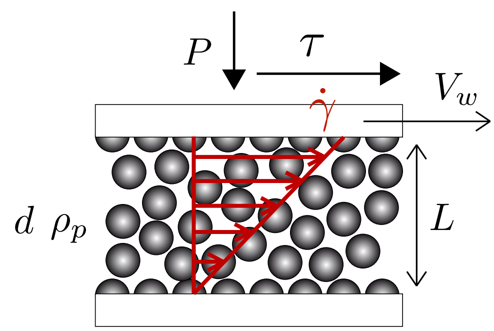



Figure 4: Plane shear at constant pressure.

A crucial observation raised by Da cruz *et al.* [18, 19] and by Lois *et al.* [64] is that, in the simple sheared configuration for infinitely rigid particles, dimensional analysis strongly constrains the stress/shear rate relations [36]. For large systems ($L/d >> 1$), and for rigid particles (the young modulus being much higher than the confining pressure), the system is controlled by a single dimensionless parameter called the inertial number:

$$I = \frac{\dot{\gamma} d}{\sqrt{P/\rho_p}}. \tag{1}$$

As a consequence, dimensional analysis imposes that the volume fraction $\phi$ is a function of $I$ only, and that the shear stress $\tau$ has to be proportional to the normal stress $P$, which is the only stress scale of the problem. The constitutive laws can then be written as follows:

$$\tau = P\mu(I) \quad \text{and} \quad \phi = \phi(I), \tag{2}$$

where $\mu(I)$ is a friction coefficient, which depends on the inertial number. The shape of the friction coefficient $\mu(I)$ and of the volume fraction $\phi(I)$ are provided by numerical simulations using discrete element models and by experimental measurements. The Figure 5 presents a summary of results coming from different studies for 2D systems (disks) or 3D (spheres). One observes that the friction coefficient $\mu$ is an increasing function of the inertial number. Friction increases when increasing the shear rate and/or decreasing the pressure. In the limit of quasi-static flows ($I -> 0$)

the friction coefficient tend towards a constant. The volume fraction also varies with $I$. It start at a maximum value when $(I->0)$ and decreases more or less linearly with $I$. It is interesting to note that in the range of inertial number corresponding to the dense flow regime, the macroscopic friction coefficient $\mu(I)$ and the volume fraction $\phi(I)$ do not depend on microscopic properties of grains. Changing the coefficient of restitution of the grains, or changing the inter-particle friction coefficient (as long as it is not zero), does not change the macroscopic friction [19].

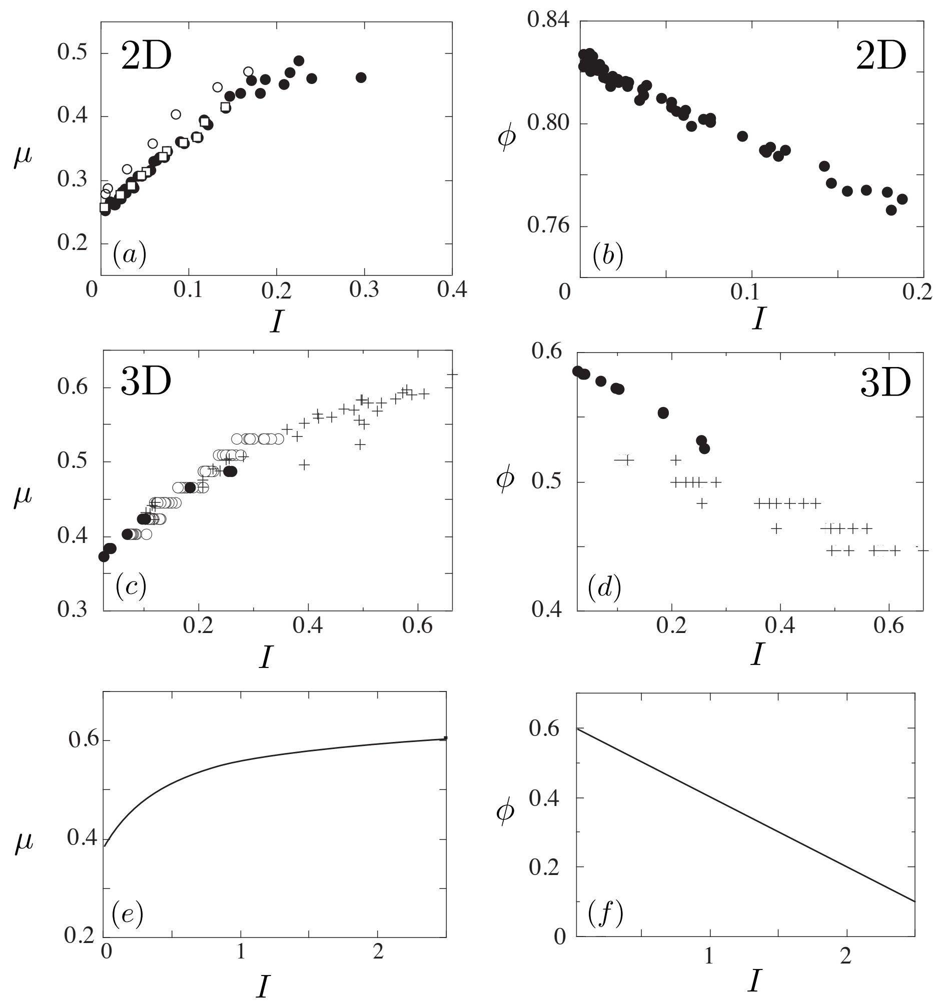



Figure 5: Friction law $\mu(I)$ and volume fraction law $(\phi(I))$; (a) (b) for 2D configurations with disks (c) (d) for 3D configurations with spheres; (e) (f) empirical analytical law proposed (eqs. (3)) data fraom [80, 96, 64, 102].

The inertial number appears to be the important parameters controlling the rheology of dense granular flows. It can be interpreted in terms of the ratio between two time scales: a microscopic time scale $d/\sqrt{P/\rho_p}$, which represents the time it takes for a particle to fall in a hole of size $d$ under the pressure $P$, and which gives the typical timescale of rearrangements ; and a macroscopic timescale $1/\dot{\gamma}$ linked to the mean deformation. This interpretation allows to classify more precisely the different flow regimes. Small $I$ correspond to a quasi-static regime in the sense that macroscopic deformation is slow compared to microscopic rearrangements, whereas large values of $I$ correspond to rapid flows. The dimensional analysis tell us that, to switch from quasi-static to inertial regime, one can either increase the shear rate or decrease the pressure. This inertial number is also equivalent to the square root of the Savage number or Coulomb number introduced by some authors as the ratio of collisional stress to total stress [96, 1].

In the plane shear configuration, the velocity profile is linear. It is then tempting to assume that relations (2) obtained in this configuration give the intrinsic rheology of the granular medium. This is true only if the stresses that develop in an inhomogeneous flow are the same as in the plane shear. It is the case if the rheology is local, namely, if the stresses depend only on the local shear rate and on the local pressure. Under the assumption of a local rheology one can then use eqs (2) as constitutive equations. Fitting the experiments and numerical simulations, it is possible to propose analytical expressions for the friction law and the volume fraction law, which can then be used to study other configurations. An example of phenomenological expressions are:

$$\mu(I) = \mu_1 + \frac{\mu_2 - \mu_1}{I_0/I + 1} \qquad \text{and} \qquad \phi = \phi_{max} + (\phi_{min} - \phi_{max})I. \tag{3}$$

Typical values of the constants obtained for monodispersed glass beads in 3D are: $\mu_1 = \tan 21^o$, $\mu_2 = \tan 33^o$, $I_0 = 0.3$, $\phi_{max} = 0.6$, $\phi_{min} = 0.4$. Those functional forms have not been tested for large values of the inertial number $I$. However, the choice of a friction law that saturates to a finite value $\mu_2$ when $I$ goes to infinity is supported by experiments of steady granular fronts flowing down a slope [81]: at the tip of a front the shear rate goes to infinity, whereas experiments reveal that the slope, and hence the friction coefficient, remains finite. This is consistent with the saturation of $\mu(I)$ to $\mu_2$.

Before discussing applications of this simple phenomenological description of granular flows, it is important to say that eq. (2) can be generalised to a tensorial form. When written in terms of scalar like in 2, the rheology can only described flows sheared in a single direction. In order to describe more complex 3D configurations the friction law has to be written in term of the shear rate tensor [51]. The simplest way to do so, is to assume that the flow is incompressible, i.e we neglect variation of volume fraction and that the pressure is isotropic. We also assumed that the shear stress tensor is colinear to the shear rate tensor, as proposed by previous authors [37, 94] and as suggested by numerical simulations [23]. The stress tensor can then be written in terms of an effective viscosity as follows:

$$\sigma_{ij} = -P\delta_{ij} + \tau_{ij} \tag{4}$$

where $P$ is the isotropic pressure,

$$\tau_{ij} = \eta_{eff}\dot{\gamma}_{ij}, \qquad \text{with} \qquad \eta_{eff} = \frac{\mu(I)P}{|\dot{\gamma}|}, \tag{5}$$

and where $|\dot{\gamma}|$ is second invariant of the shear rate tensor: $|\dot{\gamma}| = \sqrt{\frac{1}{2}\dot{\gamma}_{ij}\dot{\gamma}_{ij}}$.

Within this description, the granular liquid is described as an incompressible non newtonian fluid, with an effective viscosity $\eta_{eff} = \mu(I)P/|\dot{\gamma}|$. This viscosity diverges when the shear rate $|\dot{\gamma}|$ goes to zero, which insures the existence of a flow threshold given by :

$$|\tau| > \mu_1 P \quad \text{with} \quad |\tau| = \sqrt{\frac{1}{2}\tau_{ij}\tau_{ij}}. \tag{6}$$

This description is very similar to the one developed in other visco-plastic material like mud. However, granular matter is peculiar, because the viscosity depends on the pressure and not only on the shear rate.

In the following, we show that this approach predicts some important features of granular flows.

### 2.2 Example of application of the local rheology

#### 2.2.1 Flow down inclined plane

Let us consider a granular layer flowing down a rough inclined plane (Fig. 6*a*). We first consider the steady and uniform regime. The stress distribution in this configuration is given by $\sigma_{xz} = \rho g \sin\theta(h-z)$ and $\sigma_{zz} = P = -\rho g \cos\theta(h-z)$. One can then apply the constitutive law 3 to predict the velocity profile $u(z)$ and the volume fraction profile $\phi(z)$. The ratio between shear and normal stress being constant, one obtains the following relation:

$$\mu(I) = \tan\theta \quad \text{with} \quad I = \frac{d\, du/dz}{\sqrt{g\phi\cos\theta(h-z)}} \tag{7}$$

This equation implies that the inertial number is constant across the layer, which directly implies that the volume fraction $\phi$ is independent of $z$. The velocity profile can also be integrated, assuming that the roughness condition corresponds to a zero velocity at the base. The predicted velocity varies like $z^{3/2}$ and is called a Bagnold profile [6, 101] :

$$\frac{u(z)}{\sqrt{gd}} = \frac{2}{3} I_0 \frac{\tan\theta - \mu_1}{\mu_2 - \tan\theta}\sqrt{\phi\cos\theta}\left(\frac{h^{3/2} - (h-z)^{3/2}}{d^{3/2}}\right). \tag{8}$$

These predictions can be compared with experiments and numerical simulations. The first comparison concerns the flow threshold. According to eq. (8) a steady and uniform flow is possible only if the inclination is comprised in between a minimum angle $\theta_1 = \arctan\mu_1$ and a maximum angle $\theta_2 = \arctan\mu_2$. This is observed in experiments, in which steady uniform flows are observed only in a range of inclination [80]. However, contrary to the prediction, the minimum angle to get a flow is not a constant, but depends on the thickness of the layer (Fig. 6*c*). This critical angle $\theta_{stop}(h)$ is higher for thin layers than for thick layers, which is a signature of non local effects. The second comparison concerns the shape of velocity and volume fraction profile. For thick layer, numerical simulations show that the flow is well described by the Bagnold profile (Fig. 6*b*). The agreement is less good for thin layers, where the profile becomes more linear [88]. The constant volume fraction profile is also observed in the simulation. A last prediction concerns the evolution of the mean velocity. Experiments [80] and simulations [102] shows that there exist a correlation between the depth averaged velocity $\bar{u}$, the thickness of the layer $h$ and the inclination angle $\theta$:

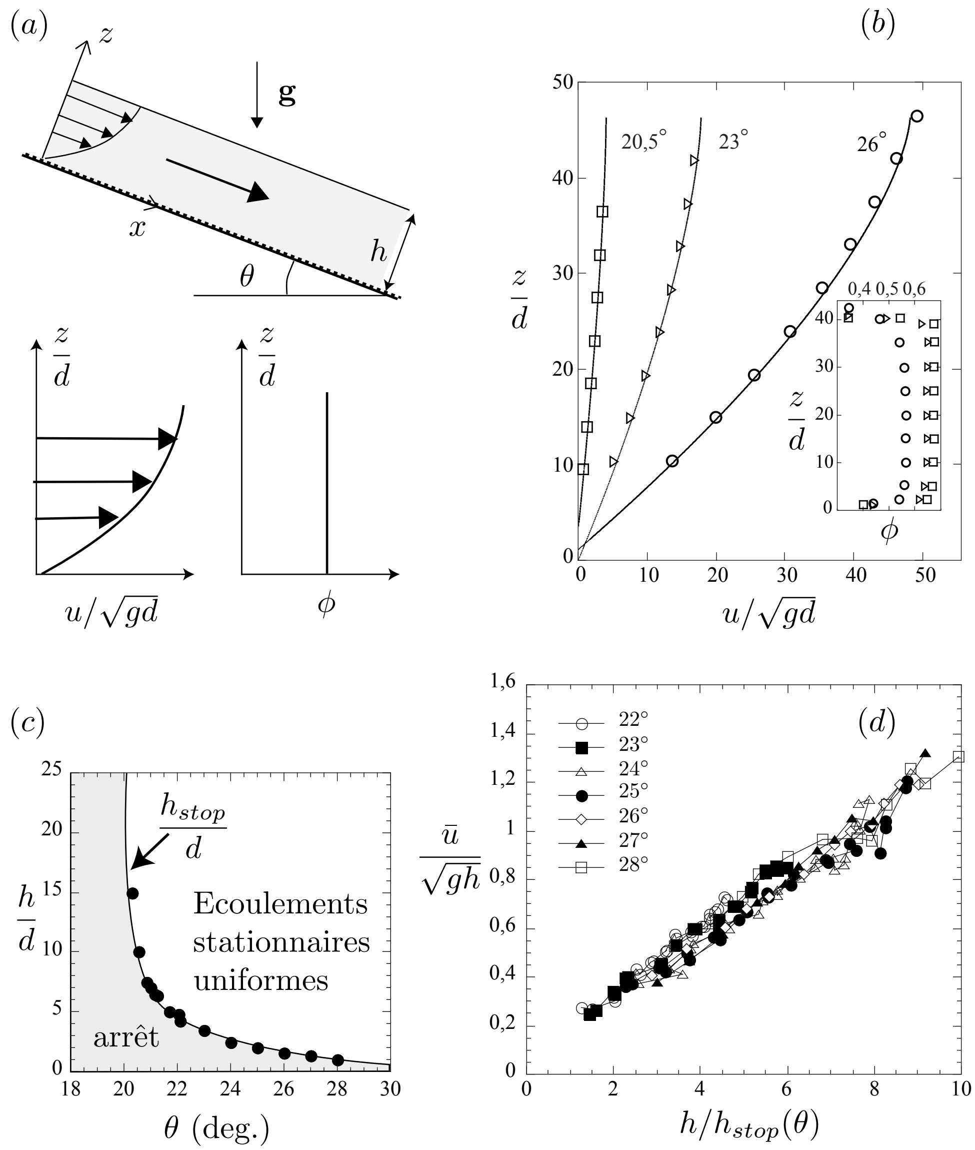



Figure 6: (a) Flow down inclined planes and the prediction of the local rheology; (b) Comparison between the Bagnold velocity profile (lines) and molecular simulation for spheres. Inset, volume fraction profile (data from [9]). (c) domain of existence of steady uniform flows. (d) Normalised depth averaged velocity as function of $(h/h_{stop})$ (from [80]).

$$\frac{\bar{u}}{\sqrt{gh}} = \beta \frac{h}{h_{stop}}, \tag{9}$$

with $\beta \simeq 0.14$ is a constant and $h_{stop}(\theta) = \theta_{stop}^{-1}$ is the minimal thickness necessary to get a flow at inclination $\theta$. The scaling of $\bar{u}$ with $h^{3/2}$ is compatible with the prediction of the local rheology $\mu(I)$. However, the angle dependence suggests a relation between the function $h_{stop}(\theta)$ and the friction law $\mu(I)$ [36]. Whether this link is a coincidence or reveals a more profound physical meaning remains an open question [27].

The local rheology is then able to capture some characteristics of the steady uniform flows down inclined planes. One can go one step further and analyse the stability of such thin flows. It is well known with classical fluids that, when the flow becomes faster and faster, the free surface eventually becomes unstable and presents long wave modulations [93]. This instability is called the Kapitza instability for viscous liquids, roll waves instability for turbulent flows, and is also observed with mud [7]. It has been shown that the same kind of free surface deformation is also observed with granular materials (Fig. 7 [32]). The granular roll waves have been experimentally investigated and the dispersion relation of the instability, namely how the small perturbations are amplified or attenuated depending on their frequency, has been measured using a well controlled forcing method at the entrance of the flow [32]. These measurements provide a severe test for rheological models, as the characteristics of the instability strongly depend on the rheological properties of the liquid. In order to test the relevance of the local visco-plastic rheology, Forterre [33] has performed the linear stability analysis of the problem, using the tensorial formulation of the friction law eq. (4). He has shown that once the parameters of the friction law are calibrated using the steady uniform flows, the theory gives quantitative predictions for the instability threshold and the dispersion relation of the instability (Fig. 7). This study shows that the proposed local rheology is relevant to describe non trivial three dimensional flows.

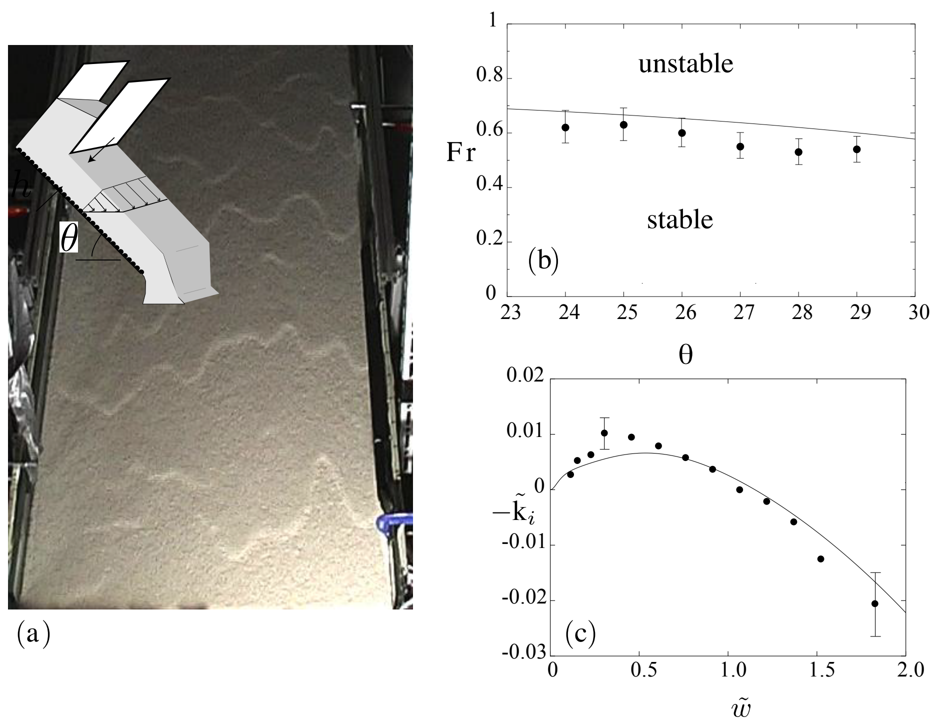



Figure 7: (a) picture showing the free surface waves observed when a granular media flows down an inclined plane. (b) Instability threshold in terms of the Froude number versus the inclination. lines: prediction of the local rheology, markers, experimental measurements. (c) spatial growth rate versus frequency (from [33]).

### 2.2.2 Flow on a pile

Another interesting configuration to study the granular rheology is the flow on a pile, obtained when a granular layer flows on a static heap (Fig. 8). Contrary the case of the flow down an inclined plane, the inclination $\theta$ and the thickness $h$ of the flow are not imposed by the experimentalist, but are selected by the system itself. The only control parameter is the flow rate $Q$. In this configuration, it has been shown that steady and uniform flows are possible if the system is confined in between two walls. In this case, the additional friction induced by the lateral walls plays a crucial role and is responsible for the localisation of the flow at the free surface [104, 50]. One can try to model this configuration using the visco-plastic description, by taking into account the lateral boundary conditions. In the case of lateral rough walls, a typical velocity profile predicted by the theory is plotted in Fig. 8$a$. The model correctly predicts the shape of the profile with a localisation close to the free surface. The agreement is not only qualitative, but quantitative predictions can be made for the free surface velocity, once the friction law $\mu(I)$ is calibrated as explained in the previous section (Fig. 8$a$) [51]. This is a second example of a flow with a shear deformation in different directions, which is well described within the framework of a local rheology. However, the description is not perfect and some experimental observations are not well captured by the model. First a transition from a continuous flow regime to an avalanching regime is observed when the flow rate decreases [62, 51]. This transition is not predicted by the model. Secondly, the interface between the flowing region and the static pile is not as discontinuous as predicted by the theory. Experimentally, a slow creep is observed in the static region, with an exponential tail which is not predicted by the local rheology [56].

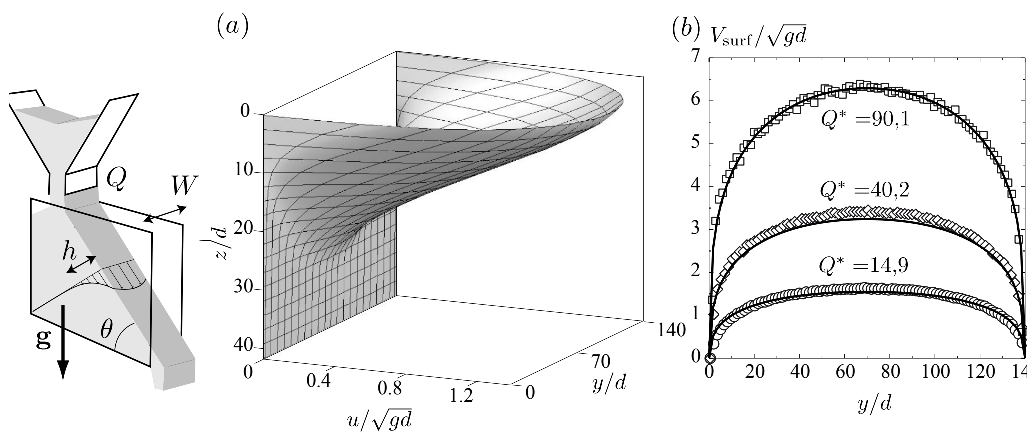



Figure 8: Flow on pile. (a) 3D velocity profile predicted by the visco-plastic local rheology. (b) quantative comparison between the theory and experiments for the velocity profile observed at the free surface (From [51]).

### 2.2.3 Granular collapse

A flow configuration, which has attracted a lot of attention the last ten years, is the collapse of a granular column under gravity. A cylinder full of grains is suddenly lift up. The material then spreads over the surface. This configuration is a model for cliff collapses in geophysics [59, 68, 8]. Experiments have revealed interesting scaling for the spreading distance as a function of the aspect ratio of the initial column. However a complete description is still lacking. An interesting question is whereas the local visco-plastic approach would be able to correctly predict the dynamics of this fully three dimensional flow. However, implementing the rheology in three dimentional fluid mechanics code is a tricky work [35], which to our knowledge is not yet completly achieved. However, a recent study performed by Lacaze and Kerswell [58] suggests that the approach may be relevant to describe the whole dynamics. This authors have performed numerical simulations of the collapse problem using molecular dynamics simulations. Knowing at each time, the position of the particles, their velocities, the forces at each contact, their are able by a suitable coarse graining process to compute the shear rate, the shear stress, the pressure, and check at each position, at each time, how the friction coefficient varies with the inertial number. Fig. 9 shows that all the points collapse quite well along a line, which has the same shape as the one obtained in simple configurations as plane shear. This result gives good hope that the visco-plastic description is enough to capture most of the dynamics of this complex three dimensional flows.

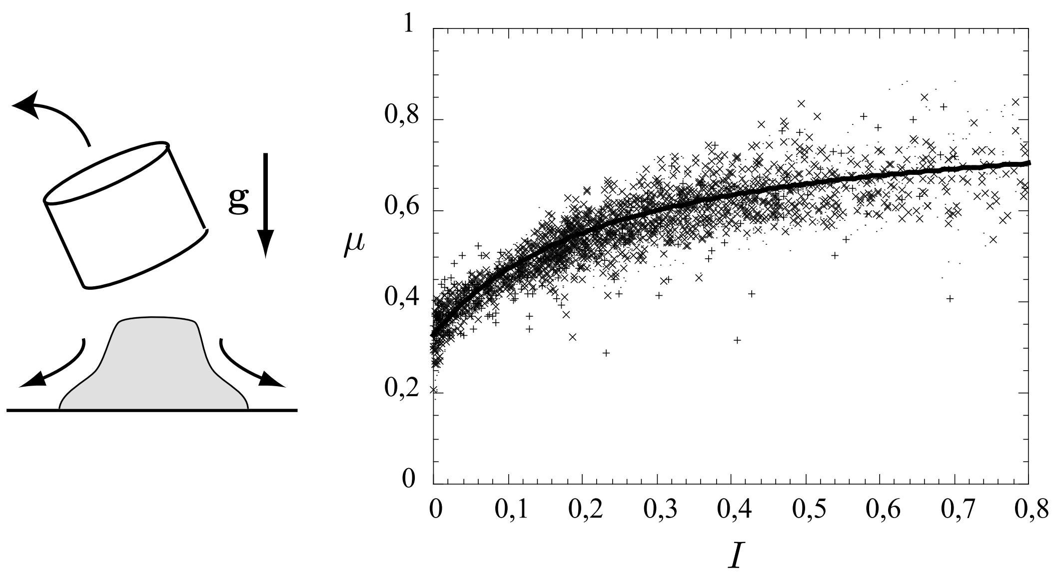



Figure 9: Collapse of a granular column under gravity simulated using molecular dynamics. Each point corresponds to the ratio $\mu = \tau/P$ as a function of $I$ for different positions, different times, different aspect ratio. The continuous curve is the best fit (from [58]).

#### 2.2.4 Confined flows

The flows considered in the previous sections are free surface flows. Other geometries, in which the material is confined in between walls, have been also intensively studied, including the cylindrical Couette cell [106](Figure 3*b*), the vertical silo (Figure 3*c*), and the plane shear with gravity ([10, 36] and reference therein). In all these configurations, the velocity profile is localized in a shear band 5 to 10 particles thick located close to the moving wall. In more complex 3D geometries such as the modified Couette cell, where the bottom is split in a rotating and a static part, shear zones up to 40 particle diameters wide are observed [30]. It is important to keep in mind that all these flows are most of the time conducted in a quasi-static regime, for which the inertial number is less than $10^{-4}$. In this regime, the $\mu(I)$ rheology reduces to a simple frictional Drucker-Prager plasticity criterion (eq. (6)) and does not correctly predict the shear bands observed experimentally. The localization of the shear close to the moving wall is predicted, as it is due to a non uniform stress distribution. However, the thickness of the predicted shear bands depends on the shear velocity and vanishes in the quasi-static limit. This is in contradiction with the observations and clearly shows that the local rheology is not able to capture the quasi-static regime.

Although the $\mu(I)$ rheology does not correctly predict the width of the shear bands, it can be useful to predict their position in cases where a complex 3D pattern develops. For example, in the case of the flow induces by the rotation of a disk in a granular media, the shear band takes the form of a cap (Fig. 10), or a column. The local rheology captures the correct shape and the transition between a cap and a column state, depending on the aspect ratio. In this example the viscous part of the visco-plastic rheology does not play any role, but allows to approach this quasi-static problem from a fluid mechanics point of view, which appears to be easier than from a pure plasticity point of view.

### 2.3 Beyond $\mu(I)$

The previous examples have shown that the phenomenological rheology $\mu(I)$ allows to describe many properties of dense granular flows. However, we have seen that some characteristics are not predicted by this simple approach. In this section we discuss the different limits of the local approach and the other theoretical attempts developed to describe granular flows.

#### 2.3.1 The solid-liquid transition: Role of the preparation, hysteresis, and finite size effects

The first limit of the $\mu(I)$ rheology concerns the starting and stopping properties. Within the model, the flow threshold is described by a unique friction angle $\mu_1$, which corresponds to a simple Coulomb criterion. However, the transition between flow and no flow in a granular medium is a more complex phenomenon.

First, the way the material starts depends on the initial preparation of the sample. Both the initial volume fraction and the history of the previous deformation play a role [21]. In order to describe these effects, it is necessary to introduce additional internal variables, like the volume fraction, the texture, which characterize the anisotropy of the force network. Attempts exist within plasticity models [92], but the link with the $\mu(I)$ rheology remains to be done.

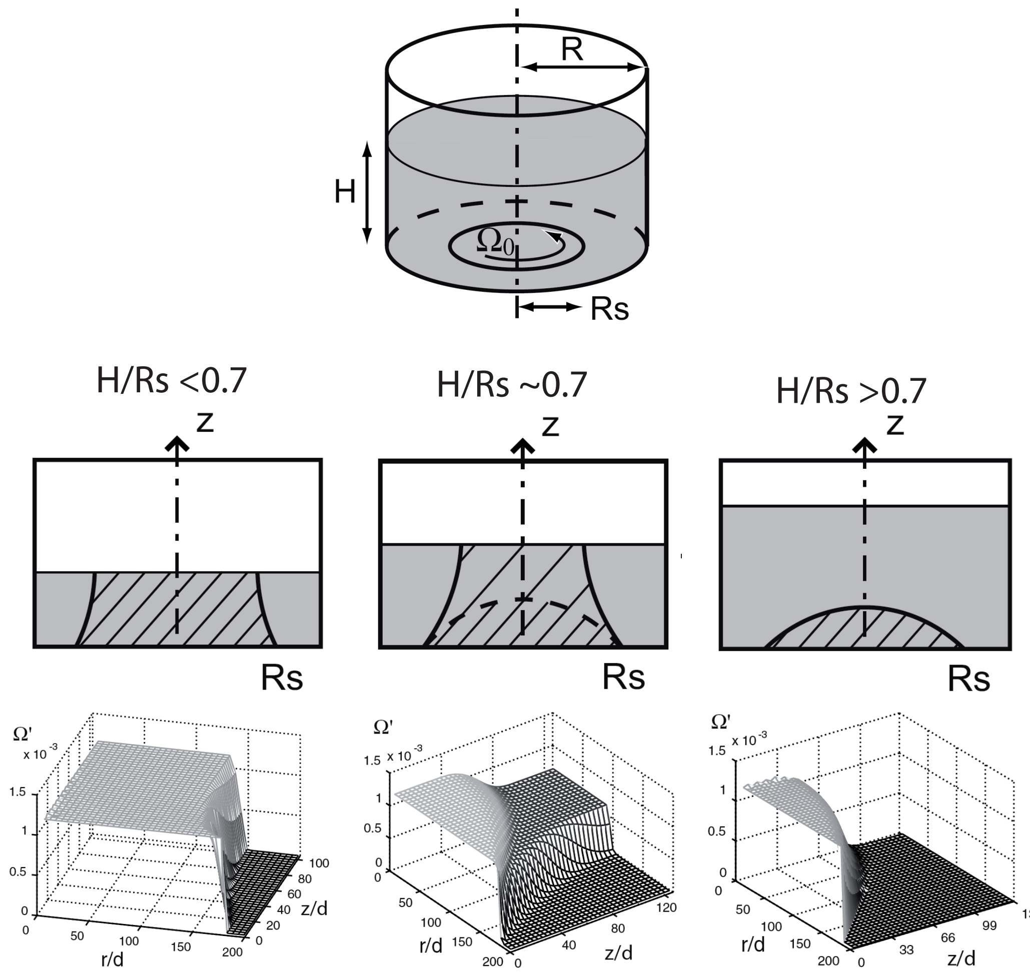



Figure 10: Shear bands created by a split bottom container. Depending on the aspect ratio, the granular media entrained by the disk take different forms, well predicted by the simulation of the local rheology (bottom line) (figure from [52]).

A second limit of a simple Coulomb criterion is that it does not describe the hysteresis observed in some flow configurations. For example, let us consider the case of a granular layer on a rough inclined plane. Starting from a static layer of thickness $h$, one has to incline the plane up to a critical angle $\theta_{start}$ in order for the flow to start. Once the layer moves, one has to decrease the inclination below $\theta_{stop}$ less than $\theta_{start}$ in order to stop the flow [79]. In between this two angles, the system is metastable: a small perturbation can be enough to trigger an avalanche. Depending on the inclination, the avalanche can propagate down the slope only or can go uphill and put into motion the whole layer [20] (Fig. 11*a*). This kind of hysteresis is also observed in other configurations when the system is driven by the stress, for example in a Couette cell when imposing a torque on the inner cylinder [36]. The physical origin of the hysteresis is not clear, although an analysis based on the dynamics of a single grain on a bumped surface shows that it is related to the balance between the external stress, the dissipation due to collision and the geometrical traps formed by the bump [86, 2]. A phenomenological theoretical approach has been developed

[107, 4] to describe this hysteresis. The granular media is described as a mixture of solid and liquid, which proportion is controlled by an order parameter.

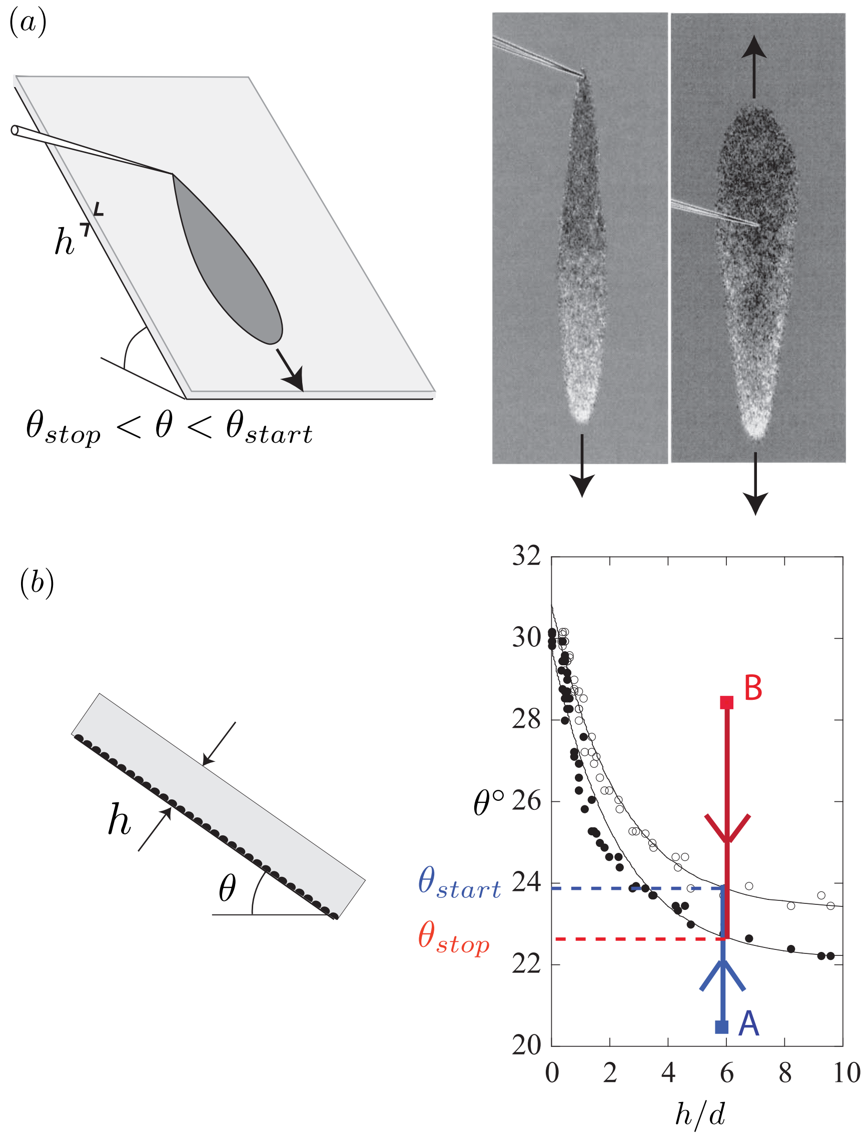



Figure 11: (a) Avalanches on a thin layer of grains initially static on a rough plane inclined in a metastable state. Depending on the inclination angle, the avalanche, which is triggered by a tiny perturbation propagates downhill only or uphill (from [20]). (b) starting and stopping angles of a granular layer on a rough inclined for glass beads (from [31]).

The last limit of the simple Coulomb description of the solid-liquid transition of a granular medium concerns fine size effects. In the case of the inclined plane for example, the starting angle $\theta_{start}$ and stopping angle $\theta_{stop}$ depend on the thickness of the layer $h$, as shown in Fig. 11*b* [79, 80]. For thick layers larger than 20 particle diameters, these angles are independent of $h$. However, for thin layers, both angles increases. This additional rigidity of a thin layer compare to a thick layer is not yet understood, but could be related to non trivial collective effects.

#### 2.3.2 Quasi-static flows

The second serious limit of the local rhelogy $\mu(I)$ concerns the description of quasi-static flows. We have seen that for confined flows in a quasi-static regime the rheology correctly captures the location of the shear bands but fails in predicting their thickness, which goes to zero when $I$ goes to zero, in contradiction with the observations. For flow on a heap also, far from the free surface, the rheology predicts a zero velocity with a true solid part, whereas experimentally an exponential tail is measured, corresponding to a slow creep motion. These observations of a slow creep motion on typically 10 particle diameters suggests that the simple assumption of a local rheology, i.e. a one to one relation between the shear rate and the stresses, is wrong. Several approaches have been proposed to described this quasi-static flows. The first consists in modifying plasticity models to take into account fluctuations of stresses [53], or rotation [73]. A second approach consists in writing non local rheological laws [72, 85]. One idea which seems to emerges from the different attemps to describe this regime is the role played by the mechanical noise and correlation [11, 84, 87]. In all the athermal systems like foams, glasses, granular systems, a rearrangement somewhere induces stress and strain fluctuations, which can in turn influence deformation somewhere else. How to take into account such non local effects in constitutive equations remains an open question.

#### 2.3.3 Transition liquid/gaz and link with the kinetic theory

A last limit of the local rheology concerns the transition towards the gaseous regime describe by the kinetic theory of granular gases [14, 38, 46, 47, 69]. This transition is much less studied than the solid-liquid transition, although it can be observed in several configurations [64, 65]. For example, it can be observed when a flows layer on a very steep plane. It does not reach a steady regime but accelerates and becomes more and more dilute [31]. In flow on a heap confined between two walls also, if the flow rate is too large, a gaseous layer develops at the free surface [67, 50]. This transition to a very dilute regime is not predicted by the simple $\mu(I)$ approach, but it is well described by the kinetic theory of granular flows. However, the kinetic theory does not predict the correct behaviour in the dense flow regime, and predicts a friction coefficient $\mu$ which decreases with the inertial number [34]. Trying to reconciliate the two approaches has motivated several theoretical work trying to modify the standard kinetic theory of granular gases to be compatible with dense flow regimes [49, 66, 48, 57].

#### 2.3.4 Conclusion about the local rheology

We have shown that, to the first order, the behaviours of a dense granular flow can be described using simple dimensional arguments and the assumption of a local rheology. Within this approach, the granular medium is described as frictional visco-plastic liquid, with a friction coefficient depending on the shear rate. This phenomenological approach predicts with success many flow configurations. However, serious limits exist concerning the transition to the static regime or the rapid regime.

Morevover, most of the studies deal with spherical particles and the generalisation to more complex material media is another open question.

We show in the next section that the lack of more precise information about the rheology can in some cases be circumvented by writing depth averaged conservation equations. By depth averaging, it is no longer necessary to specify a bulk rheology of the material, an expression for the basal stress being sufficient. These depth averaged approach is very useful in many geophysical contexts.

## 3 Depth averaged approach

Depth-averaged or Saint-Venant equations have been introduced in the context of granular flows by Savage & Hutter [98]. The initial motivation was to model natural hazards such as landslides or debris flows [75, 42]. Assuming that the flow is incompressible and that the spatial variation of the flow takes place on a scale larger than the flow thickness, one obtains the Saint-Venant equations by integrating the three-dimensional mass and momentum conservation equations. For two-dimensional flows down a slope making an angle $\theta$ with the horizontal (see Figure 12), the depth-averaged equations reduce to

$$\frac{\partial h}{\partial t} + \frac{\partial h\langle u\rangle}{\partial x} = 0, \tag{10}$$

$$\rho_s\phi\left(\frac{\partial h\langle u\rangle}{\partial t} + \alpha\frac{\partial h\langle u\rangle^2}{\partial x}\right) = \left(\tan\theta - \mu_b - K\frac{\partial h}{\partial x}\right)\rho_s\phi g h\cos\theta, \tag{11}$$

where $h$ is the local flow thickness, $\langle u\rangle = Q/h$ is the depth-averaged velocity ($Q$ being the flow rate per unit of width), and $\phi$ is the volume fraction, assumed constant.

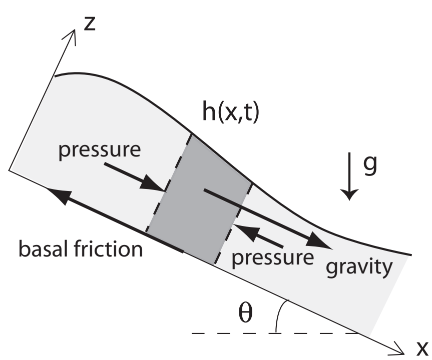



Figure 12: Forces balance in shallow water description.

Equation (10) is the mass conservation and Equation (11) is the momentum equation, where the acceleration is balanced by three forces (Figure 12): the gravity parallel to the plane, the tangential stress between the fixed bottom and the flowing layer (written as a basal friction coefficient $\mu_b$ times the normal stress), and a pressure force related to the thickness gradient. The coefficient $\alpha$ is related to the assumed velocity profile across the layer and is of order 1. The coefficient $K$ represents the ratio of the normal horizontal stress ($x$-direction) to the normal vertical stress ($z$-direction) and is close to one for steady uniform flows [101]. The main advantage of the Saint-Venant equations is that the dynamics of the flowing layer can be predicted without knowing in details the internal structure of the flow. The complex three-dimensional rheology of the material is mainly embedded in the basal friction term $\mu_b$.

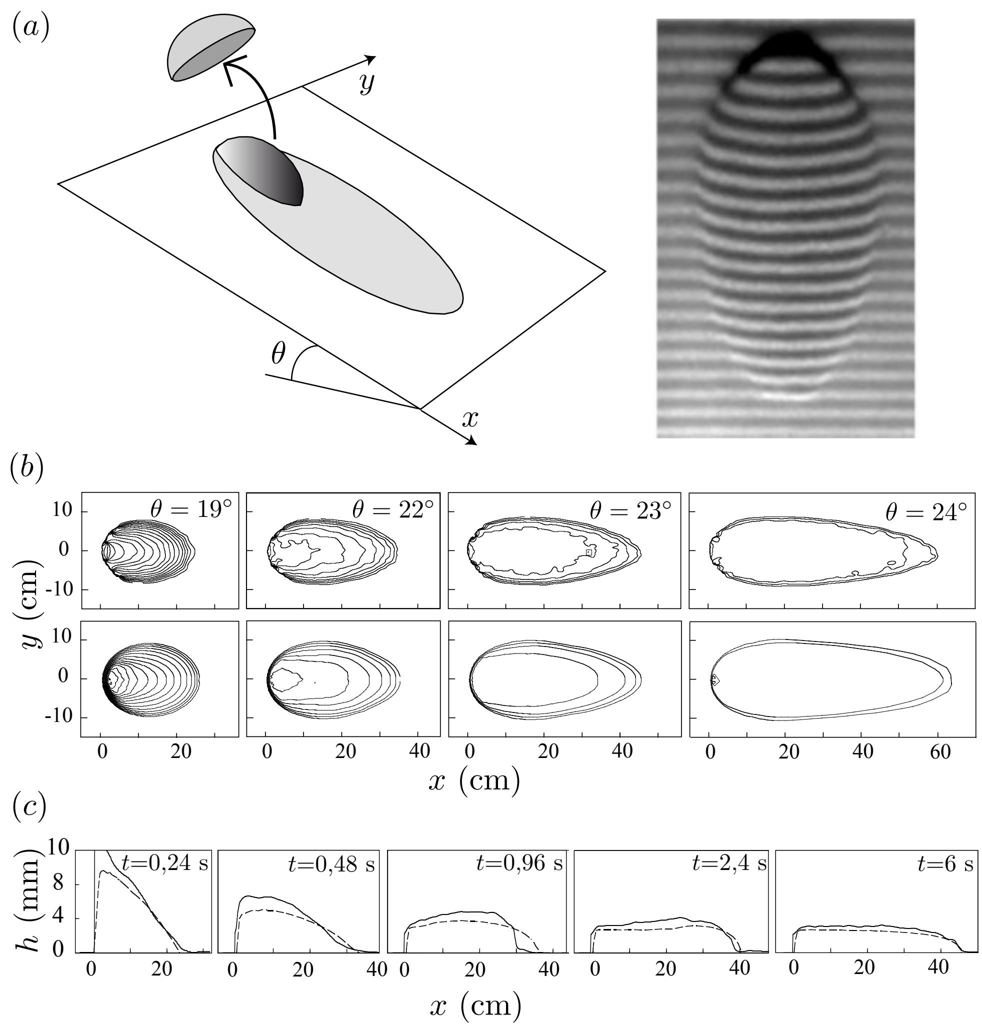



Figure 13: (a) spreading of a granular mass down an inclined plane measured by a Moir (picture). (b)FInal deposit as a function of the inclination. Comparison between experiment (top raw) and depth averaged equations (bottom raw). (c) Spreading dynamics as a function of time: experiment (solid line) and theory (broken line).

Taking a simple constant Coulomb-like basal friction is sometimes sufficient to capture the main flow characteristics [98] and has been used to describe granular slumping [8, 60, 70], rapid flows down smooth inclines [41, 108], and shock waves [40, 43]. However, for flows down rough inclines, the assumption of a constant solid friction is not compatible with the observation of steady uniform flows over a range of inclination angles. One can then use the local rheology developed in the previous section to propose an expression for the basal friction. In order to properly capture the hysteresis and the influence of finite size, more complex basal friction laws

$\mu_b(\langle u \rangle, h)$ have been proposed, which lead to quantitative predictions in complex situations such as a propagating steady front, mass spreading, or surface instabilities [81, 82, 32, 71] (Fig. 13). It should be noticed that the Saint-Venant Equations (11) and (12) represent a first order development in terms of the flow aspect ratio. Therefore, they do not capture second-order effects like longitudinal and lateral momentum diffusion, which stabilize instabilities [33] and control lateral stresses. The knowledge of the full 3D constitutive equations (eq (4)) may allow the development of more complex depth-averaged models [93, 7].

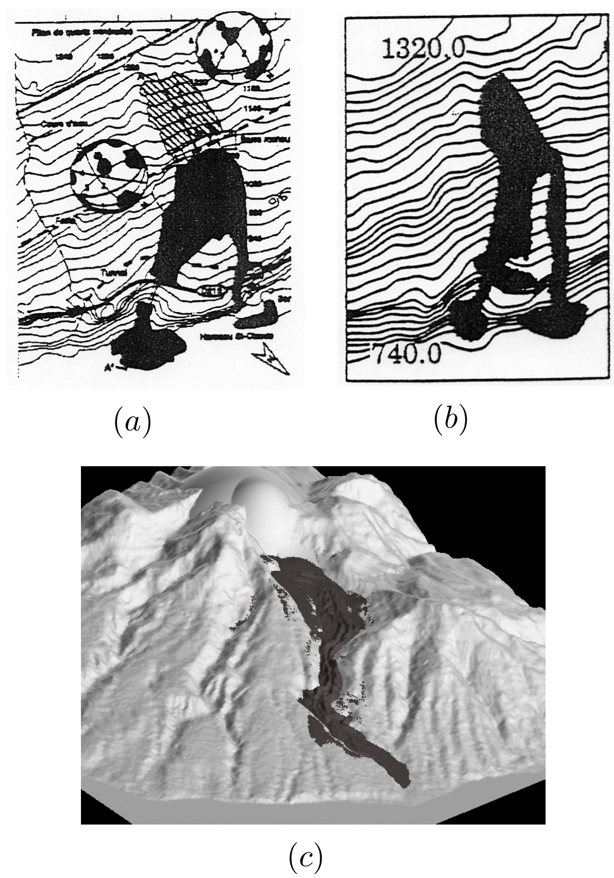



Figure 14: Applications of depth averaged equations to natural events (a) trajectory observed for the land slide of Charmonetiers, Isere 1987 and simulation (b) from [75]. (b)simulation of the boxing day event (December 26th 1997), Montserrat Island, Lesser antilles (from [42]).

Another application of the depth-averaged equations concerns situations where the flowing layer propagates on an erodible layer, such as flows on top of a static pile. In this case,an exchange of matter exists between the liquid and solid phase. An additional equation is then needed to determine the solid-liquid interface. Several closures have been proposed [3]. The first model [12, 13] assumes that erosion/deposition is controlled by the difference between the local slope and the critical pile angle. Other approaches assume a relation between the averaged velocity and the flow thickness, either by fixing the velocity gradient [25] or by prescribing a basal shear stress at the solid/liquid boundary [54]. These models predict qualitatively non trivial behaviours such as the propagation of avalanche fronts [26, 105]. Although these two-layer approaches seem a promising framework to study avalanching flows on erodible beds, it is important to note that the closures proposed to date are not compatible with observations of steady uniforms flow on pile, where the flowing thickness is selected by the side walls [50]. This clearly shows that a proper development of shallow water models has to rely on the knowledge of the full constitutive equations, a goal not yet completely achieved.

To conclude this section on depth averaged equations, it is important to emphasize that this framework is often used to model real situations encountered in geophysics. It is possible to take into account a complex topography by considering an inclination which varies with space and by adding additional centrifugal forces [39]. Examples of simulation of rock avalanches and pyroclastic flows are shown in Fig. 14.

## 4 Towards more complex granular materials

We have seen that in the case of a simple granular medium, some advances have been made in the description of the flow properties. In this section we discuss to which extend this progress in our understanding of granular flows can help in describing more complex material encountered in applications. We discuss the role of the polydispersity, the role of cohesion and the role of the presence of an interstitial liquid.

### 4.1 Polydispersed media

Most of the granular media encountered in applications are made of grains having different sizes. A major problem, which arises when manipulating polydispersed material is the size segregation [76, 97, 29]. During the flow, particles of different sizes tend to separate. Despite its importance in industrial processes, and despite the large number of studies, the mechanisms responsible for segregation are far from being understood. Different scenarii have been proposed (percolation, statistic sieving [97]). A review of the research on segregatation is far beyond the scope of this paper. In this section, we will focus on the rheology of polydispersed material and discuss how the friction law evidenced in monodispersed granular media can be modified to account for the presence of different grains.

For monodispersed material the rheology is given by a friction coefficient depending on the inertial number $I = \dot\gamma d/\sqrt{P/\rho_p}$, where $d$ is the particle diameter. It is then tempting to generalize this approach to the case of polydispersed material by using the local mean particle diameter $\bar d$ in the definition of $I$:

$$I_{\bar d} = \frac{\dot\gamma \bar d}{\sqrt{P/\rho_p}} \tag{12}$$

This idea has been recently tested for the flow of disks down inclined planes [90]. In this study, the material is composed of two different sizes: large disks of diameter $d_b$ and small ones of diameter $d_s$. As for monodispersed material, there exists a range of inclination for which a steady uniform flow develops. Segregation induces a non uniform distribution of large particles across the layer: the large are at the free surface, whereas the small concentrate at the bottom. The authors introduced a local mean diameter defined by :

$$\bar{d}(z) = \frac{\phi_s(s)d_s + \phi_b(z)d_b}{\phi(z)} \tag{13}$$

where $\phi$ is the volume fraction of the medium, and $\phi_s$ and $\phi_b$ are the local volume fraction of small and big particles respectively ($\phi = \phi_s + \phi_b$). The first results obtained by Rognon et al [90] is that the inertial number computed using this mean diameter is constant across the layer in agreement with the local rheology. Consequently, if $I_{\bar{d}}$ is constant, this implies that $\dot{\gamma} \propto 1/\bar{d}$ for a given inclination and position. The velocity gradient is then inversely proportional to the mean size of the particles. This prediction is observed in the simulation: the shear rate is higher at the bottom where there is an accumulation of small particles, and decreases at the top where large particles migrates. This result strongly suggests that the frictional visco-plastic rheology may be relevant for polydispersed material. However, if the rheology tell us how the change in relative concentration of species changes the flows, it does not predict the segregation leading to the distribution of grains. This point remains a challenge in the physics of granular media.

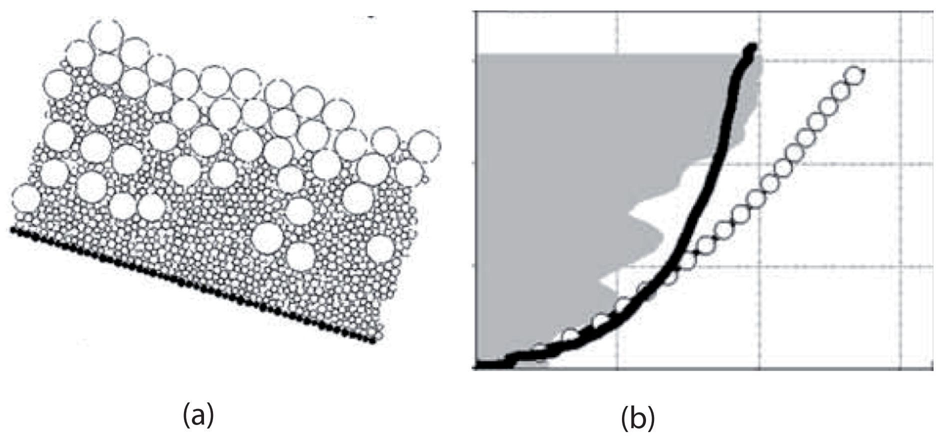



Figure 15: Flow of bidispersed material on an inclined plane in 2D (a) Sketch of the flow. (b) Velocity profile (solid line) compare to velocity profile observed in the monodispersed case (symbols). The shaded region corresponds to the concentration of large particles (from [90]).

### 4.2 Cohesive granular media

The local rheology can also be generalized to the case of cohesive material. In this case one can consider that the grains interact not only by contact interaction but that an additional attractive force exists, which tends to put the grains in contact. The origin of cohesion can be the Van der Waals interactions, capillary bridges, electrostatic forces... Taking into account in the rheology the details of these cohesive interactions is still an open question. However, some authors have studied a simpler case [89], in which the force interaction is simply characterized by a maximum force $N_c$. The interaction force is then zero if the particles are not in contact, decreases to a maximum $-N_c$ and increases again when approaching the grains further. The force then becomes again positive i.e. repulsive, when the overlap between particles becomes large (one recover the repulsive elastic interaction) (Fig. 16). This model represents the simplest cohesive material and allows to capture the major features of the flow of cohesive granular media.

To study the rheology of this simple cohesive material, one can consider the plane shear configuration (Fig. 4) where the material is sheared at a constant shear rate $\dot{\gamma}$ and confined under a pressure $P$. This analysis has been carried out by Rognon *et al* [89] using molecular dynamics simulation. In this case also, the dimensional analysis of the problem appear to be very fruitful. By contrast with the dry case for which the inertial number $I = \dot{\gamma}d/\sqrt{P/\rho_p}$ was the single dimensionless control parameter of the problem, the case of the cohesive material introduces a second dimensionless number $C$, which is the ratio of the maximum attractive force $N_c$ divided by the characteristic pressure force $Pd^2$ ($Pd$ in 2D):

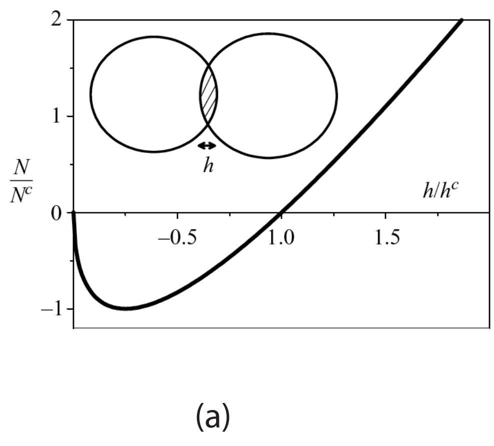
$$C = \frac{N_c}{Pd^2} \tag{14}$$

One can then directly conclude that the friction coefficient and the volume fraction can be written as follows:

$$\mu = \mu(I, C) \quad \text{and} \quad \phi = \phi(I, C) \tag{15}$$

Rognon *et al* [89] have systematically studied the variation of $\mu$ and $\phi$ with $I$ and $C$ (Fig. 16).

Although this study concerns an oversimplified model of cohesive material, it suggests that the progress made in our understanding of the rheology of dry granular materials can serve as a base to developed rheological models for cohesive material.



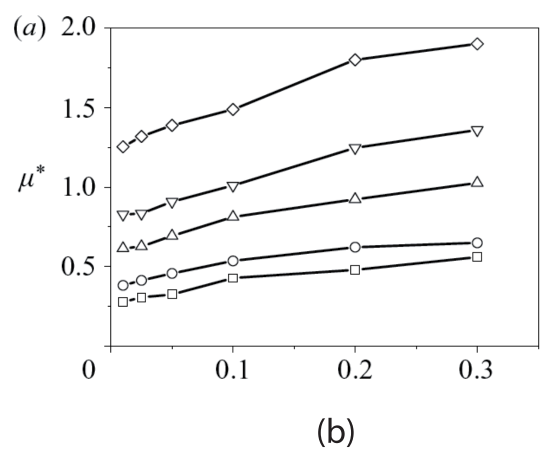




Figure 16: (a) Interaction force version overlap between particles for the model of cohesive material studied in [89].(b) Friction coefficient as a function of the inertial number for different value of the cohesion parameter $C$ (C= 0, 10, 30, 50, 70 from top to bottom) (from [89]).

### 4.3 Immersed granular media

The last example we would like to discuss concerns the case of granular media immersed in a liquid. To which extend the presence of the liquid modifies the rheology of the material, and is it possible to propose constitutive equations for the granular paste. A simple way to address this question consists again in considering the plane shear configuration under constant pressure. The grains immersed in a liquid are confined by a porous plate which applied a pressure $P^p$ on the particle and imposed a shear rate $\dot{\gamma}$. As in the dry case, we want to know how the shear stress $\tau^p$ and the volume fraction varies with $\dot{\gamma}$ and $P^p$. We have seen that the important idea is to compare the typical time of deformation $t_{macro} = 1/\dot{\gamma}$ and the typical time of rearrangement $t_{micro}$. If the deformation is slow compared to the typical time it takes for a particle to fall in a hole, it can be considered as a quasi-static deformation. This picture suggests a first naive approach of the rheology of immersed granular media. The presence of the fluid is going to change the typical falling time of a grains $t_{micro}$ and will then change the constitutive law of the material.

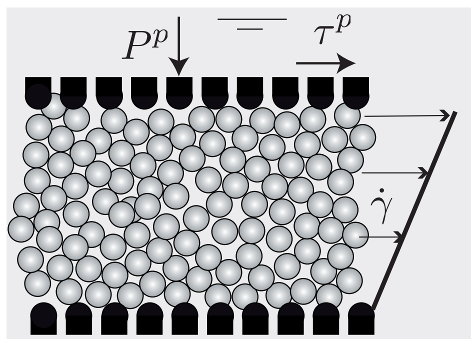



Figure 17: The plane shear configuration under constant normal stress for an immersed granular material

The study of the time taken by a particle to fall in a fluid under a pressure $P_p$ has been done in [16, 15] and has put in evidence different regimes. The equation controlling the motion of the particle of mass $m$ can be written as follows:

$$m\frac{d^2z}{dt^2} \simeq P^p d^2 - F_{drag} \tag{16}$$

One can then distinguish between three regimes:

- *Free fall regime*

  In this regime the drag induced by the fluid is negligible during the fall. The particle during its motion follows an accelerated motion described by the two first terms of eq. (16). This is the dry regime discussed before. Considering that $z \simeq d$ and $t \simeq t_{micro}$ in the first term, on get $t_{micro}^{fall} = d/\sqrt{P^p/\rho_p}$

- *viscous regime*

  In this regime the grain rapidly reaches it terminal velocity given by the balance between the viscous drag and the pressure. Knowing that $F_{drag} \simeq \eta d\, dz/dt$ one find that $t_{micro}^{visc} \simeq \eta/P^p$

- *inertial regime*

In this regime the grain also reaches its terminal velocity but it is controlled by the inertial drag force given by $F_{drag} \simeq C_d d^2 \rho_f (dz/dt)^2$ where $C_d$ is the drag coefficient. On then get that: $t^{inert}_{micro} \simeq d/\sqrt{P^p/(\rho_f C_d)}$.

The transition between the different regimes is then controlled by two dimensionless numbers: a Stockes number $St$, which is the ratio between the free fall time scale over the viscous time scale, and the number $r$ ratio of the free fall time over the inertial time scale

$$St = \frac{t^{fall}_{micro}}{t^{visc}_{micro}} \simeq \frac{d\sqrt{\rho_p P^p}}{\eta} \tag{17}$$

$$r = \frac{t^{fall}_{micro}}{t^{inert}_{micro}} \simeq \frac{\rho_p}{\rho_f C_d} \tag{18}$$

A phase diagram can then be drawn in the parameters space ($St$, $r$). If the longest time is $t^{fall}_{micro}$ (if $St >> 1$ and $r >> 1$), one get the regime of dry granular flows, for which the fluid is negligible. If the longest time is $t^{visc}_{micro}$,(if $St << 1$ and $r >> St$) the regime is a viscous regime. Finally, if the longest time is $t^{inert}_{micro}$ (if $St << r$ and $r << 1$), the regime is inertial.

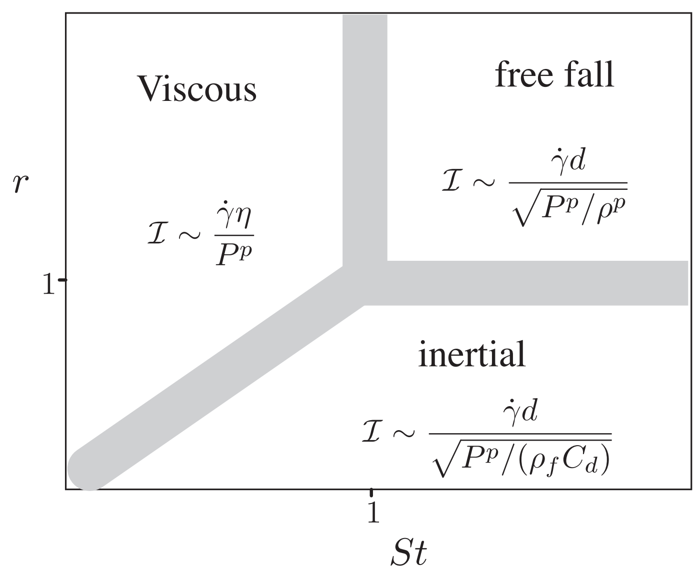



Figure 18: The different flow regimes in the plane ($St$,$r$) for immersed granular flows sheared under a confining pressure $P^p$. The expression of the dimensionless number $\mathcal{I}$ is given in each regime.

When coming back to the problem of the plane shear configuration, one can then suggest that the relevant dimensionless number is going to be given by the shear rate multiply by the microscopic time scale. We then call this number $\mathcal{I}$. The different expressions of $\mathcal{I}$ for the three regimes are presented in Fig. 18.

From this analysis one can then proposed that the constitutive laws describing the rheology of immersed granular material are given by the friction law and the volume fraction law:

$$\tau^p = \mu(\mathcal{I})P^p \quad \text{and} \quad \phi = \phi(\mathcal{I}) \tag{19}$$

One recovers the rheology of dry granular material in the free fall regime, but other scaling laws are obtained in the others regimes. To our knowledge, there is no precise test of such a phase diagram based on the analysis of the typical time scales, and no direct measurement of the function $\mu(\mathcal{I})$ and $\phi(\mathcal{I})$ are available (except in the dry regime as discussed previously).

However, in the viscous regime, indirect measurements exist derived from experiments of flows down inclined plane. These measurements show that the friction law as a function of the viscous $\mathcal{I}$ number follows a shape similar to the one observed in the dry case [15, 78]. This approach has been successfully applied to describe the immersed flow down a pile confined between lateral walls [24].

Up to now, we have discussed the rheology of immersed granular media sheared under a constant pressure, by analogy with the dry granular case. This configuration is relevant for free surface flows like submarine avalanches, in which the gravity prescribes the stress. However, this situation is not conventional in the field of the rheology of suspensions. Most of the studies concern shear at constant volume fraction, in a situation where particles have the same density as the fluid [103]. This is equivalent to keep the top plate at a fixed distance in Fig. 17. In this case, the pressure $P_p$ on the top plate is no longer a control parameter, but has to be measured. In this configuration, the existence of different regimes has been discussed in several papers ([1, 61]). The viscous regime has attracted most of the studies on the rheology of suspensions. Under the assumption that inertia does not play any role, dimensional analysis implies that the shear stress $\tau_p$ and the normal stress $P_p$ measured at the top plate varies with $\eta\dot{\gamma}$ [74, 61]:

$$\tau_p = f_1(\phi)\eta\dot{\gamma} \quad \text{et} \quad P_p = f_2(\phi)\eta\dot{\gamma} \tag{20}$$

The function $f_1(\phi)$ is the relative viscosity, which has been measured in a wide range of $\phi$. $f_1(\phi)$ increases with $\phi$ and seems to diverge at a maximum volume fraction [44]. The normal stress $f_2(\phi)$ has been much less studied. Discussing the different models and implications of such rheology for dense suspensions is far beyond the scope of this paper dedicated to granular flows. The only important point we would like to underline, is the analogy that exists between the granular approach 19 and the suspension approach 20. The two expressions are the same if one choose $\phi(I) = f_2^{-1}(1/\mathcal{I})$ and $\mu(\mathcal{I}) = \mathcal{I}f_1(f_2^{-1}(1/\mathcal{I}))$. For the two expressions to be really identical, the divergence of $f_2(\phi)$ when approaching the maximum volume fraction has to be the same as the divergence of $f_1(\phi)$, in order to have a friction coefficient $\tau^p/P^p$ going to a constant. This analogy suggests that immersed granular media and suspension could be described within the same framework.

In the simple shear cell configuration we have discussed, there is no relative motion between the fluid and the granular material. The fluid moves with the grains. However, there are configurations for which it is no longer the case. One example is the sediment transport. When a liquid flows on top of a granular bed, the fluid put into motion the grains. In this case, the drag force induced by the velocity difference between the fluid and the grains is the motor of the granular flows. Another example in which the fluid moves relatively to the grains is the initiation of avalanches. When a granular matter starts to deform, it compacts or dilates depending on the initial preparation. In presence of a liquid, a dilatation (resp. a compaction) of the granular layer means that the liquid has to flow into (resp. out of) the granular layer. This fluid motion induces an additional stress on the granular skeleton, which modifies the deformation. A framework, which seems to be relevant to address this coupling problem is the two phase flow equations framework. The idea is to consider the liquid phase and the solid phase as two continuum media, and in writing the mass and momentum conservation equations for the two phases. The challenge lies in proposing an expression for the stresses in each phases. The simple idea we have developed in this section based on the friction law and the volume fraction law (eq. (19)) gives a relevant suggestion for the rheology of the granular phase. Some recent works on sediment transport [77] and on the triggering of submarine avalanches [78] seems to show the relevance of this approach.

## 5 Conclusion

We have presented a survey of our current understanding of dense granular flows. Our main intention was to emphasize that a zero order description of the viscous-like behavior of dense granular flows is now available, which relies on simple but solid dimensional arguments. A frictional visco-plastic formulation has been developed which gives quantitative predictions for different flow configurations and can serve as a first tool to predict other configurations encountered in applications. Although promising, this approach fails to capture the details of the quasi-static flows and the transition to solid or gaseous regimes. It is difficult to anticipate that more elaborated constitutive equations will be developed in the near future that can describe the whole phenomenology of granular flows. The diversity of the theoretical approaches clearly shows that the task is difficult, the central question being, in our opinion, how to take into account non-local effects created by the network of enduring contacts.

We have also discussed in this paper how the recent progresses in our understanding of simple dry granular flows can serve as a base to tackle more complex materials as the one used in industry or encountered in geophysics. The role of cohesion, the role of polydispersity, the role of the interstitial fluid has been discussed. Fundamental research on such complex granular materials is very young, and no doubt that the intense activity will give rise in the next future to important progresses.

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