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The de Sitter and anti-de Sitter Sightseeing Tour

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1 Introduction

While celebrating the $100^{\rm th}$ anniversary of the discovery of special relativity [1], it may not be inappropriate to open a window on the de Sitter universes, as their importance in contemporary physics is gradually increasing. Just to mention two examples, the astronomical evidence for an accelerated expansion of the universe gives a central place to the de Sitter geometry in cosmology [2] while the so-called AdS/CFT correspondence [3] supports a major role for the anti-de Sitter geometry in theoretical physics.

From the geometrical viewpoint, among the cousins of Minkowski spacetime (the class of Lorentzian manifolds) de Sitter and anti-de Sitter spacetimes are its closest relatives. Indeed, like the Minkowski spacetime, they are maximally symmetric, i.e. they admit kinematical symmetry groups having ten generators[1]. Maximal symmetry also implies that the curvature is constant (zero in the Minkowski case).

Owing to their symmetry, it is possible to give a description of the de Sitter universes without using the machinery of general relativity at all. However, it is worth saying right away that, even if they share important features with Minkowski spacetime, their physical interpretation is quite different and the technical problems to be solved in order to merge de Sitter spacetimes with quantum physics are considerably harder.

The aim of this note is to give a simple and short geometrical introduction to the de Sitter and anti-de Sitter universes and to briefly comment on their physical meaning.

2 An analogy: non-Euclidean spaces of constant curvature

One easy way to replace the usual flat geometry of the Euclidean physical space $\mathbb{R}^3$ with some curved geometry consists in moving to a fictitious four-dimensional flat world and considering there the geometry of convenient three-dimensional hypersurfaces. The simplest curved model of space is the surface of a hypersphere embedded in a four-dimensional Euclidean flat space $\mathbb{R}^4$:

$$\mathbb{S}^3 = \{x \in \mathbb{R}^4,\ \ x_1^2 + x_2^2 + x_3^2 + x_4^2 = a^2\}. \tag{1}$$

$\mathbb{S}^3$ is homogeneous, isotropic and has positive curvature with value $6/a^2$. The six-dimensional invariance group of $\mathbb{S}^3$ is simply the rotation group SO(4) of the four-dimensional ambient space; it can be interpreted as the group of motions of the spherical space in the same way as the Euclidean group E(3) (translations and rotations) is the group of motions of $\mathbb{R}^3$. The main difference is that there are no commutative "translations" on $\mathbb{S}^3$.

All the non-Euclidean geometrical properties of the hypersphere come by restriction to it of the Euclidean geometry of the fictitious ambient space. In particular all geodesics, that are the analog in the curved geometry of what are straight lines in the flat case, can be obtained by intersecting the hypersphere with two-planes passing through the geometrical center of the sphere

[1] A four dimensional Riemannian manifold has an isometry group with at most ten generators. In the Minkowski case the isometry group is the Poincaré group and the ten independent transformations have a familiar physical interpretation: one time translation, three spatial translations, three rotations and three boosts.

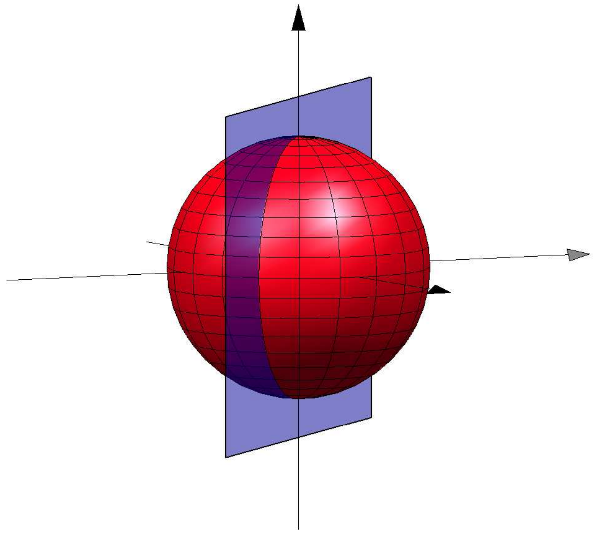

Figure 1: *A spherical model of space (positive curvature). Geodesics are maximal circles and are obtained by intersecting the sphere with two-planes passing through the center of the sphere in the ambient space.*

(see Figure 1). One recognizes immediately that in this geometry "straight lines" are maximal circles.

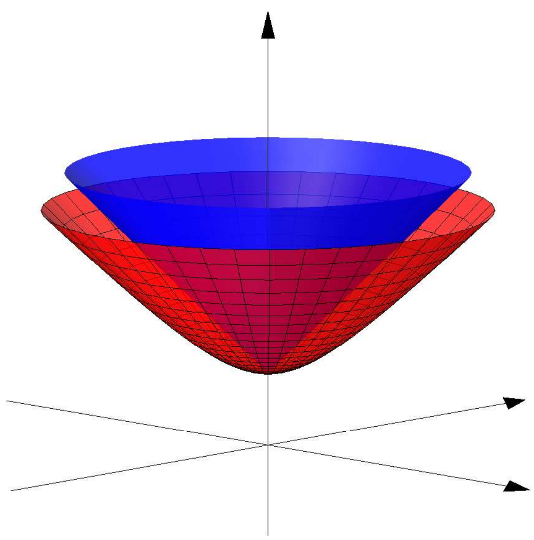

Figure 2: *A hyperbolic model of space (negative curvature).* $\mathbb{H}^3$ *(the red surface) is spacelike in the ambient Minkowski spacetime. Geodesics are branches of hyperbolae.*

The second possibility is a bit more complicated and produces a space with negative curvature. One moves again to a fictitious four-dimensional world, but now this is a four-dimensional Minkowski spacetime $\mathbb{M}^4$ (loosely speaking, a timelike direction has been added to the Euclidean $\mathbb{R}^3$, while in the previous case a spatial direction was added). Here, a model of space with negative constant curvature is the upper sheet of the two-sheeted hyperboloid $\mathbb{H}^3$:

$$\mathbb{H}^3 = \{x \in \mathbb{M}^4, \quad x_0^2 - x_1^2 - x_2^2 - x_3^2 = a^2\}. \tag{2}$$

As shown in Figure 2 the lightcone emerging from any point of $\mathbb{H}^3$ does not meet the surface anywhere else. This means that, in the ambient spacetime, the surface is *spacelike* and, as such, it is a good model for a space. As before the geometry of $\mathbb{H}^3$ is constructed by restriction of the Lorentzian geometry of the ambient Minkowski spacetime $\mathbb{M}^4$. In particular, the six-dimensional isometry group of $\mathbb{H}^3$ is the Lorentz group $\mathrm{SO}(1,3)$ of the ambient spacetime. Geodesics are branches of hyperbolae, obtained as before by intersecting $\mathbb{H}^3$ with two-planes containing the center.

3 The de Sitter universe

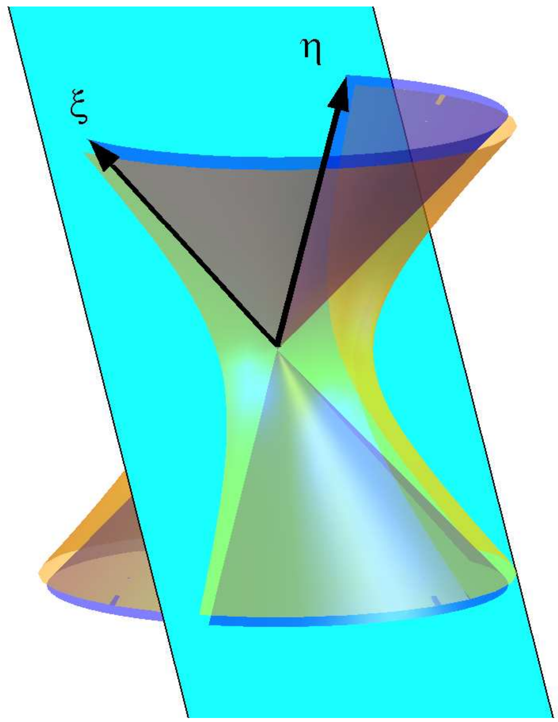

Figure 3: *The yellow surface represents the de Sitter universe. The blue cone is the lightcone of the five-dimensional ambient spacetime, asymptotic to the de Sitter hyperboloid. Timelike geodesics are hyperbolae and are obtained by intersecting the hyperboloid with two-planes passing through the origin of the the ambient spacetime. Any two-plane associated with a timelike geodesic can be identified by specifying two null vectors ξ and η that can be used also to parametrize the geodesic itself. In flat spacetime geodesics are labeled by their four-momentum. By analogy, the lightcone C can be interpreted as de Sitter momentum space. In particular, de Sitter plane waves are constructed using vectors belonging to the lightcone C [7].*

Let us now introduce a five-dimensional Minkowski spacetime $\mathbb{M}^5$ by adding a spacelike direction to $\mathbb{M}^4$ (just as we did in the spherical case). In $\mathbb{M}^5$ we consider the hypersurface with equation

$$dS_4 = \{x \in \mathbb{M}^5,\ x_0^2 - x_1^2 - x_2^2 - x_3^2 - x_4^2 = -R^2\}. \tag{3}$$

This is the de Sitter spacetime [5] (see Figures 3 and 4). It has constant negative curvature $-12/R^2$ (the sign depends on conventions) and reproduces (after a renormalization) Minkowski spacetime in the limit of zero curvature (i.e. when the radius R tends to infinity).

The causal structure of dS_4 is induced by restriction of the Lorentzian geometry of the ambient Minkowski spacetime $\mathbb{M}^5$ exactly as the geometry of the sphere was determined by the Euclidean geometry of the ambient $\mathbb{R}^4$. In particular, the de Sitter line element is obtained concretely by restricting the five-dimensional invariant interval to the manifold dS_4:

$$ds^2 = \left[(dx_0)^2 - (dx_1)^2 - (dx_2)^2 - (dx_3)^2 - (dx_4)^2\right]\Big|_{dS_4} \tag{4}$$

This line element is the most symmetrical solution of the field equations written down by Einstein wrote in 1917, where he introduced the famous cosmological constant Λ [4]. The radius R corresponding to a given value of Λ is

$$R = \sqrt{\frac{3}{\Lambda}}\,.$$

A pivotal role is played by the five-dimensional lightcone of the ambient spacetime:

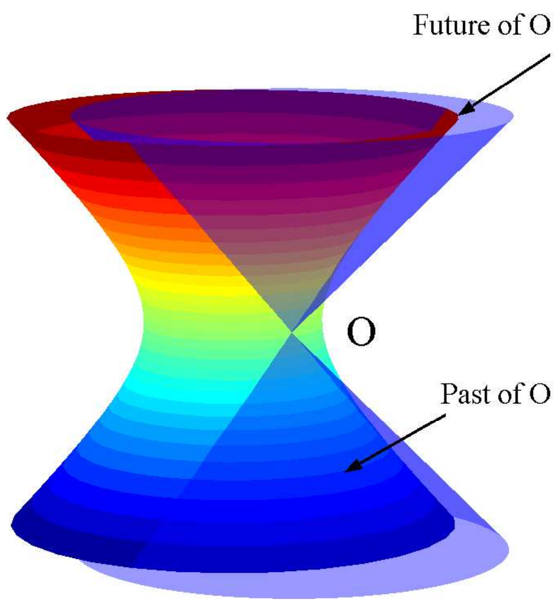

Figure 4: *The lightcone of the ambient spacetime induces the causal ordering of the de Sitter manifold. The regions shadowed by the five-dimensional lightcone emerging from the event O are the past and the future of O. In this figure the choice of colors shows a contraction era (blueshift) followed by an expansion era (redshift)*

$$C = \{\xi \in \mathbb{M}^5,\ \xi_0^2 - \xi_1^2 - \xi_2^2 - \xi_3^2 - \xi_4^2 = 0\}. \tag{5}$$

The cone C induces the causal ordering of the events on the de Sitter manifold; it also plays the role of de Sitter momentum space (see Figure 4). The de Sitter spacetime has a boundary at timelike infinity (while timelike infinity of the Minkowski manifold is a point). The cone C also provides a description of this boundary, which may be used instead of a Penrose diagram.

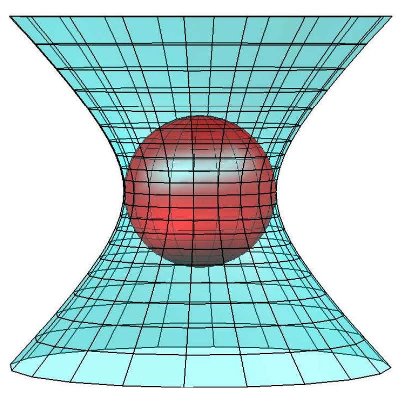

Figure 5: *A pictorial representation of the Euclidean de Sitter manifold.*

The de Sitter kinematical group coincides with the Lorentz group of the ambient spacetime $SO(1,4)$. As for the sphere, there are no commutative translations on the de Sitter manifold. This fact is source of considerable technical difficulties in the study of de Sitter Quantum Field Theory.

The relationship between the de Sitter universe and the geometry of the sphere is deeper than a mere analogy. Indeed, for imaginary times

$$x_0 \to i x_0$$

the (Euclidean) de Sitter manifold (see Figure 5) is a sphere and the Euclidean de Sitter group is the rotation group $SO(5)$. A study of the complex de Sitter manifold with applications to Quantum Field Theory has been described in [7].

The de Sitter geometry finds its most important physical applications in cosmology. In cosmology one usually "breaks" the general relativistic covariance and singles out a special coordinate system: there is a natural choice of "cosmic time" that makes the universe appear spatially homogeneous and isotropic at large scales. This property is mathematically encoded in the Friedmann-Robertson-Walker line element:

$$ds^2 = dt^2 - a(t)^2 dl^2. \tag{6}$$

The spatial distance dl^2 describes the geometry of a homogeneous and isotropic space manifold: either $\mathbb{S}^3$, $\mathbb{R}^3$ or $\mathbb{H}^3$.

In this respect the de Sitter geometry is rather special: due to the maximal symmetry and the topology of the de Sitter manifold, all three possible FRW cosmologies can be realized on de Sitter by suitable choices of the cosmic time coordinate (see Figure 6).

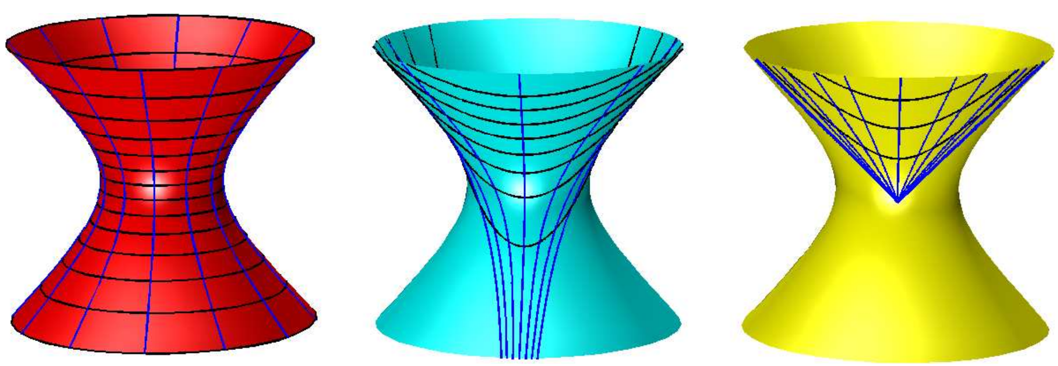

Figure 6: *Various choices of cosmological coordinates. Black curves represent hypersurfaces of constant cosmic time. Blue curves are timelike geodesics. The red manifold represents a closed FRW model with a contraction epoch followed by an expansion epoch. The light blue manifold is an exponentially expanding flat model. The yellow represents a hyperbolic open model.*

The simplest choice of time is the coordinate x_0 (see Figure 7).

$$\left\{ \begin{array}{lcl} x_0 & = & R \sinh\left(\frac{t}{R}\right) \\ x_i & = & R \cosh\left(\frac{t}{R}\right)\, \omega_i \qquad i = 1,2,3,4 \end{array} \right. \tag{7}$$

with $\omega_1^2 + \omega_2^2 + \omega_3^2 + \omega_4^2 = 1$, so that Equation (3) is easily satisfied. The hypersurfaces of constant time thus are spheres $\mathbb{S}^3$ and the coordinate system covers the whole universe. With this choice the de Sitter line element describes a closed FRW model:

$$ds^2 = (dx_0^2 - dx_1^2 - \dots dx_4^2)\big|_{dS_4} = dt^2 - R^2 \cosh^2\left(\frac{t}{R}\right) d\omega^2. \tag{8}$$

Another possible choice of time is $x_0 + x_4$ (see Figure 8). The time parameter is introduced by the relation $x_0 + x_4 = Re^{\frac{t}{R}}$; with this coordinate only one half of the manifold is covered.

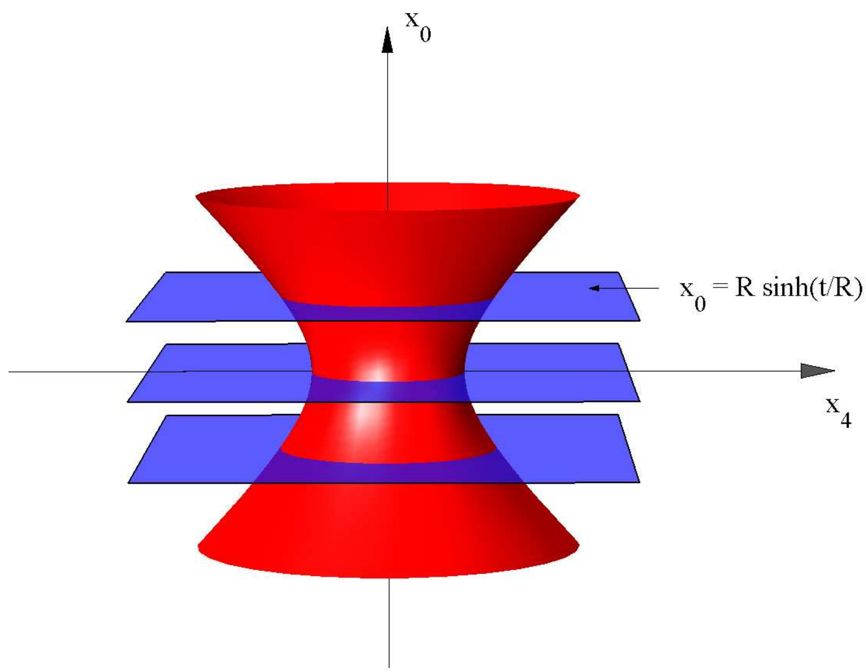

Figure 7: *Construction of the coordinate system representing the de Sitter geometry as closed FRW model. Hyperurfaces of equal cosmic time are intersection of the de Sitter manifold with hyperplanes* $x_0 = const.$

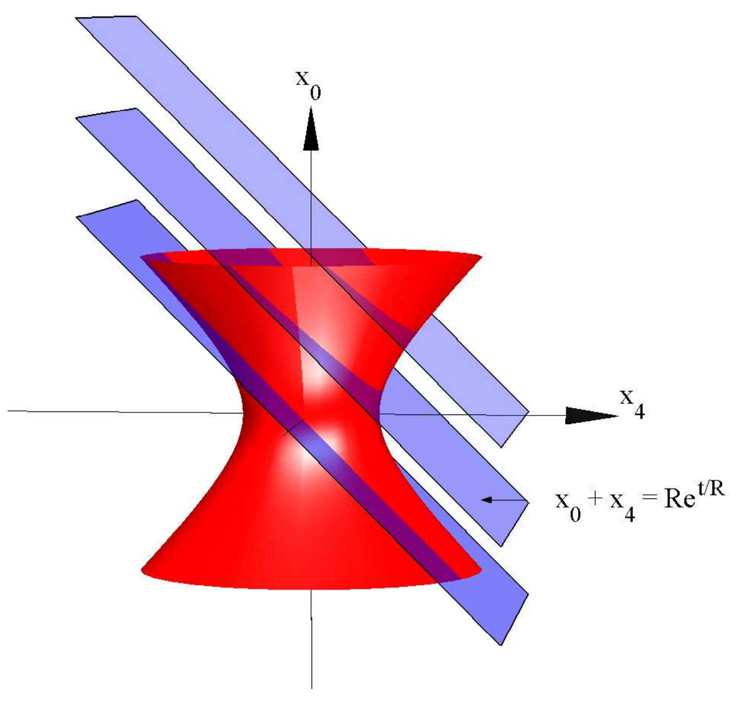

Figure 8: *Construction of the coordinate system representing the de Sitter geometry as a flat FRW model. Hyperurfaces of equal cosmic time are intersection of the de Sitter manifold with hyperplanes* $x_0 + x_4 = const.$ *Only one half of the manifold is covered since it has to be* $x_0 + x_4 > 0$.

Hypersurfaces of constant cosmic time are copies of $\mathbb{R}^3$. In these coordinates the de Sitter line element appears as a flat FRW model with exponentially growing scale factor:

$$ds^2 = dt^2 - \exp\left(\frac{2t}{R}\right) d\mathbf{x}^2. \tag{9}$$

This form of the de Sitter line element was introduced by Lemaître in 1925 [6].

It is interesting to note that the first coordinate system used by de Sitter himself was a static coordinate system with closed spatial sections. De Sitter was following Einstein's cosmological idea of a static closed universe, the idea that led to the introduction of the cosmological term in Einstein's equations. A static coordinate system (i.e. a coordinate system where nothing depends explicitly on time) is not the most natural to describe an expanding universe, but it has other interesting properties, mainly in relation to black hole physics (horizons, temperature and entropy).

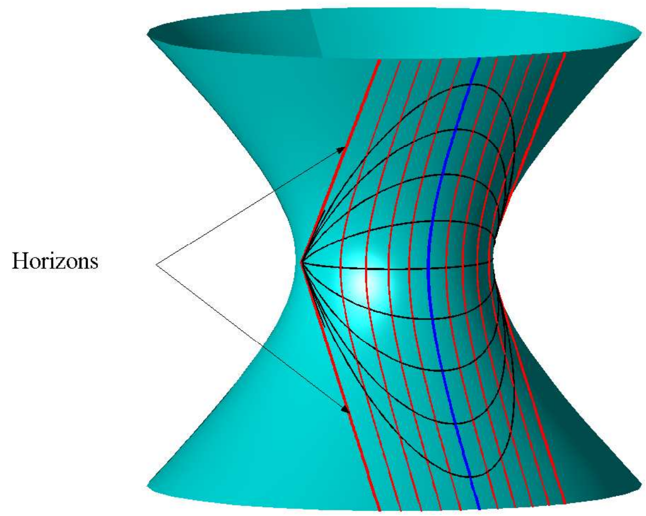

Figure 9: *A chart representing static closed coordinates. This is the coordinate system originally used by W. de Sitter in 1917. Vertical timelike curves are obtained by intersecting the hyperboloid with parallel two-planes. Only the blue hyperbola is a geodesic because it is the only one lying on a plane that contains the origin of the ambient spacetime. The other timelike curves are accelerated trajectories. They have been colored in red because there is a redshift for light sources moving along these world lines; this effect was called the de Sitter effect and was thought to have some bearing on the redshift results obtained by Slipher.*

Static closed coordinates are represented in Figure 9. The Lemaître form of the de Sitter line element is the most useful in cosmological applications. Recent observations are point towards the existence of a nonzero cosmological constant and a flat space. For an empty universe (i.e. a universe filled with a pure cosmological constant) this would correspond precisely to the above description of the de Sitter universe.

4 Anti-de Sitter

Let us now introduce a flat five-dimensional space $\mathbb{E}^{(2,3)}$ by adding a timelike direction to $\mathbb{M}^4$ (as we did in the hyperbolic case). $\mathbb{E}^{(2,3)}$ has two timelike directions and three spacelike directions and therefore it is not a spacetime in the ordinary sense (a Lorentzian manifold with one temporal and three spatial dimensions). However, the hypersurface with equation

$$AdS_4 = \{x \in \mathbb{E}^{(2,3)},\ x_0^2 - x_1^2 - x_2^2 - x_3^2 + x_4^2 = R^2\}, \tag{10}$$

is a spacetime: this is the anti-de Sitter universe (see Figure 10). It has constant positive curvature and reproduces (after a renormalization) the Minkowski spacetime in the limit when the curvature tends to zero.

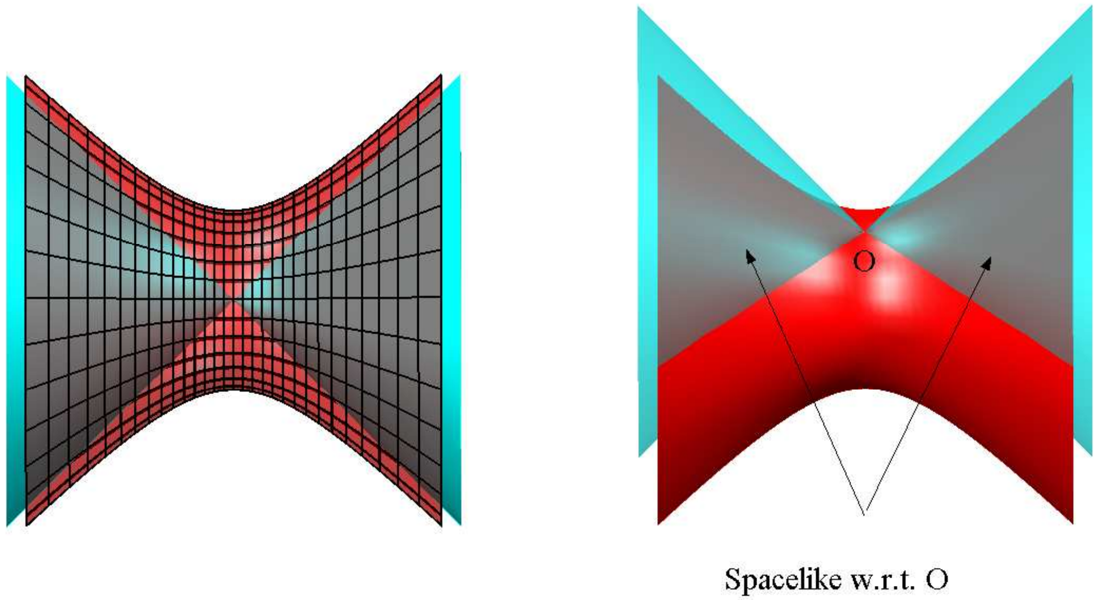

Figure 10: *A visualization of the anti-de Sitter universe. The asymptotic cone plays a crucial role exactly as in the de Sitter case. The regions of AdS_4 that are in the shadow of the five-dimensional cone emerging from an event O are the regions that are not causally connected to the event O. The asymptotic cone in the ambient space can be regarded as a representation of the boundary at spacelike infinity of the AdS manifold and carries a natural action of the conformal group that is the group-theoretical foundation for the AdS-CFT correspondence.*

The causal structure of AdS_4 is induced by restriction of the geometry of the ambient space $\mathbb{E}^{(2,3)}$ (the analogy is now with the geometry of $\mathbb{H}^3$ that is determined by the causal structure of the ambient spacetime $\mathbb{M}^4$). As before the null cone of the ambient space

$$C = \{\xi \in \mathbb{M}^5,\ \xi_0^2 - \xi_1^2 - \xi_2^2 - \xi_3^2 + \xi_4^2 = 0\} \tag{11}$$

induces the causal ordering on the anti-de Sitter manifold (see Figure 10).

Owing to the existence of closed timelike curves (see Figure 11) the causal ordering is only local. One may construct a globally causal manifold by considering the covering of the anti-de Sitter manifold (recall that the covering of a circle is a line). However even the covering of the anti-de Sitter remembers the "periodicity in time" of the original manifold: geodesics issued from an event meet again infinitely many times in the covering.

The anti-de Sitter line element is constructed by restricting the five-dimensional invariant "interval" of the ambient space to the manifold AdS_4:

$$ds^2 = \left[(dx_0)^2 - (dx_1)^2 - (dx_2)^2 - (dx_3)^2 + (dx_4)^2\right]\big|_{AdS_4} \tag{12}$$

This line element is the maximally symmetrical solution of the cosmological Einstein equations when the cosmological constant Λ is negative. The anti-de Sitter kinematical group coincides with the isometry group $\mathrm{SO}(2,3)$ of the ambient space.

The relationship between the anti-de Sitter universe and the geometry of $\mathbb{H}^3$ is deeper than a mere analogy. Indeed, for imaginary time

$$x_4 \to i x_4$$

the (Euclidean) anti-de Sitter manifold (see Figure 12) is a copy of $\mathbb{H}^4$ and the Euclidean de Sitter group is $\mathrm{SO}(1,4)$. A study of the complex anti-de Sitter manifold with applications to Quantum Field Theory has been described in [8].

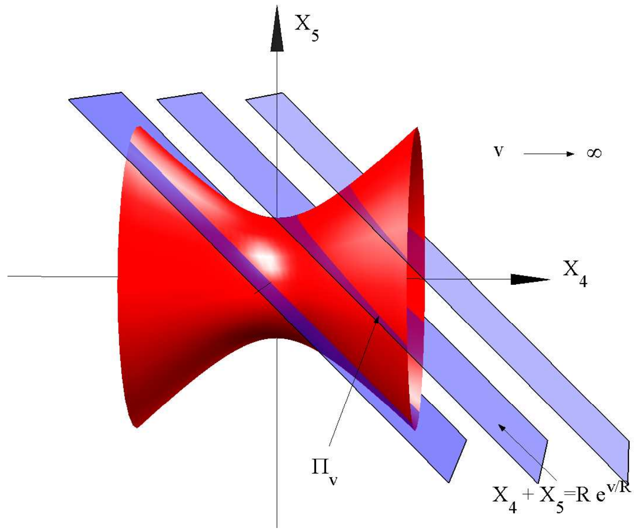

Figure 11: *Anti-de Sitter timelike geodesics are ellipses and are obtained by intersecting the hyperboloid with two-planes passing through the center of the the ambient space. The geodesics passing through a certain event all meet at the antipodal point. The focusing of geodesics remains true also in the covering space.*

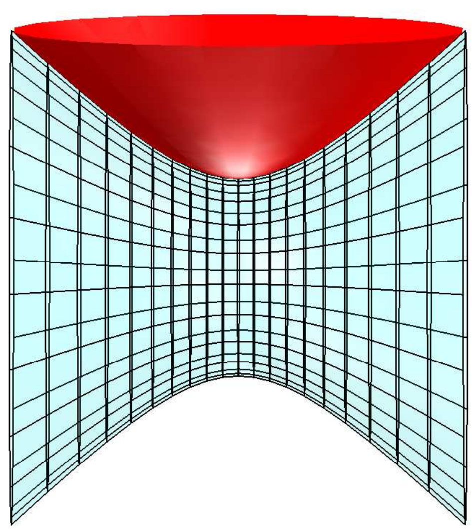

Figure 12: *Euclidean anti-de Sitter world.*

AdS is not a globally hyperbolic spacetime. In non-globally hyperbolic manifolds knowledge of equations of motion and of initial data is not enough to determine the time evolution of physical quantities. In the anti-de Sitter case, the lack of global hyperbolicity is due to the existence of a boundary at spacelike infinity: information can flow in from infinity. This fact is source of difficulties in quantizing fields on the anti-de Sitter manifolds. However this is also an opportunity since this boundary at infinity offers the very possibility for formulating the famous AdS/CFT correspondence [3].

To present an intuitive idea of this topic let us introduce coordinates on a five-dimensional anti-de Sitter manifold AdS_5 (embedded in a six-dimensional space $\mathbb{E}^{(2,4)}$) obtained by intersecting AdS_5 with hyperplanes $\{X_4 + X_5 = e^v\}$ (see Figure 13). Each slice Π_v of AdS_5 is a copy of Minkowski spacetime $\mathbb{M}^4$. Points in each slice Π_v can be thus parametrized by Minkowskian

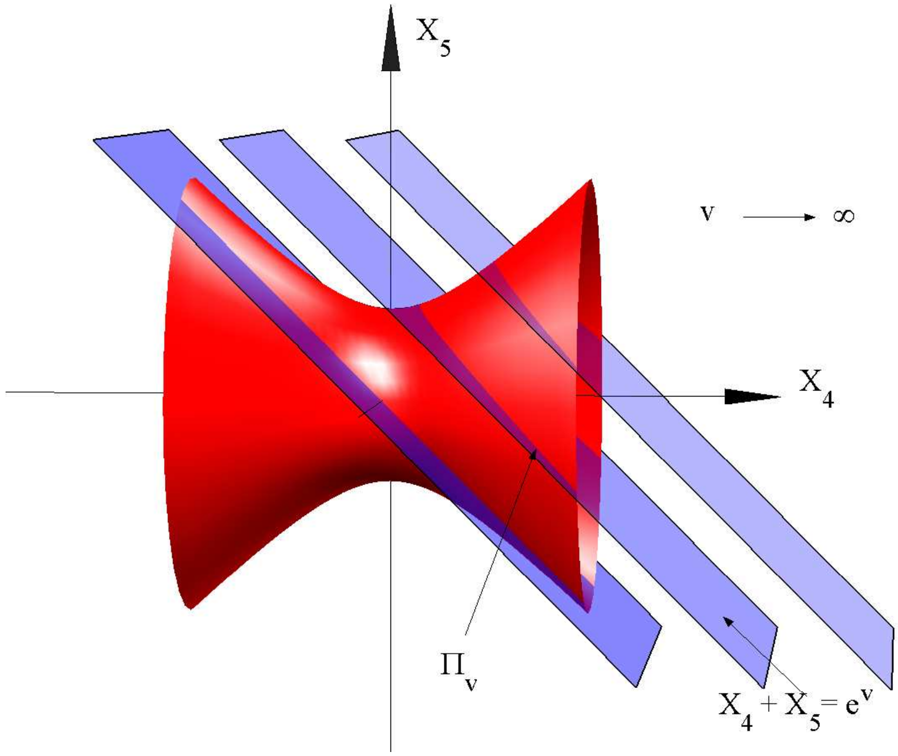

Figure 13: *Construction of the AdS-Poincaré coordinates. The limit $v \to \infty$ describes the boundary of the AdS manifold.*

coordinates x_0, x_1, x_2, x_3 (rescaled by e^v on Π_v). This explains why the anti-de Sitter coordinates (v, x_0, x_1, x_2, x_3) are also called Poincarè coordinates.

The coordinate system covers only one-half of the anti-de Sitter manifold; the anti-de Sitter metric takes the following form:

$$\begin{aligned} ds^2 &= \left[(dX_0)^2 - (dX_1)^2 - (dX_2)^2 - (dX_3)^2 - (dX_4)^2 + (dX_5)^2\right]\Big|_{AdS_5} \\ &= e^{2v}(dx_0^2 - dx_1^2 - dx_2^2 - dx_3^2) - \mathrm{d}v^2. \end{aligned} \tag{13}$$

The use of this parametrization is crucial in a recent approach to the mass hierarchy problem [9] and to multidimensional cosmology. In this context the slices Π_v are called branes. The Minkowskian geometry of the brane is induced by the ambient anti-de Sitter metric: for instance space-like separation in any slice Π_v can be understood equivalently in the Minkowskian sense of the slice itself or in the sense of the ambient anti-de Sitter universe.

When we consider the limit $v \to \infty$ we arrive at the anti-de Sitter boundary at spacelike infinity, which therefore may (essentially) be thought as a four-dimensional Minkowski spacetime. The AdS-CFT correspondence establishes an equivalence between a theory on the five-dimensional AdS_5 and a relativistic theory on the boundary $\mathbb{M}^4$ (this is an instance of another popular idea in contemporary theoretical physics: the *holographic principle*). The theory on the boundary is conjectured to have a larger symmetry group, namely the conformal group [3, 10, 11].

5 Epilogue

The de Sitter and anti-de Sitter tour now comes to its end. Before concluding let us summarize the highlights to be retained.

De Sitter's geometry is the vacuum solution of Einstein's equations with a cosmological term and plays in contemporary physical cosmology a very important double role. First, the unifying aspect of the different inflation models consists in the fact that the primordial universe has undergone a phase of exponential expansion, approximately described by de Sitter's geometry. A possible theoretical understanding of the structure of the universe which is observable today is based on de Sitter geometry at the inflation epoch.

The second motivation of interest of de Sitter geometry lies in the observational data of the recent years, starting from the observations of distant type Ia supernovae and up to the data on the temperature fluctuations of the cosmic background radiation.These observations have upturned consolidated ideas, indicating that the gravitational effect of the greatest part of the energy content of the universe is similar to Einstein's cosmological constant. This form of energy is called "dark energy".

Thus de Sitter geometry seems to assume the role of reference geometry of the universe. In other words, it seems that it is de Sitter, and not Minkowski, the geometry of spacetime deprived of its content of matter and radiation (if one describes dark energy with a cosmological constant).

Once one admits the possible existence of a cosmological constant, it is also interesting to explore the consequences of a model in which the latter is negative.

In this case spacetime geometry is termed Anti-de Sitter. This geometry has strange properties which are in confilict with common sense, as the existence of closed timelike curves and of a boundary at spacelike infinity. Nonetheless, anti-de Sitter h plays a central role in contemporary high energy physics with the formulation of the conjecture on the correspondence AdS/CFT (Anti-de Sitter/Conformal Field Theory).

In conclusion, there are still lots of rooms left in de Sitter worlds!

References

[1] A. Einstein, "Zur Elektrodynamik bewegter Körper", Annalen der Physik, **17**, 891, (1905).

[2] N. Straumann, "On the cosmological constant problems and the astronomical evidence for a homogeneous energy density with negative pressure," arXiv:astro-ph/0203330. In: B. Duplantier and V. Rivasseau (Eds): "Vacuum energy, renormalization", 7-51. Basel: Birkhäuser (2003).

[3] J. M. Maldacena, "Black Holes and Holography in String Theory". In: B. Duplantier and V. Rivasseau (Eds): "Seminaire Poincaré", 61-67. (2004). "The Large N Limit Of Superconformal Field Theories And Supergravity", Adv. Theor. Math. Phys. 2 (1998) 231, hep-th/9711200.

[4] A. Einstein, "Kosmologische Betrachtungen zur allgemeinen Relativittstheorie" *Sitzungsber. Preuß. Akad. Wiss. Berlin*, 142 (1917).

[5] W. de Sitter, "On the Curvature of Space", Proc. Kon. Ned. Akad. Wet. 20, 229 (1917).

[6] C. Møller,"The theory of Relativity". Oxford: Oxford University Press (1951).

[7] J. Bros and U. Moschella, "Two-point Functions and Quantum Fields in de Sitter Universe," Rev. Math. Phys. **8** (1996) 327. J. Bros, H. Epstein and U. Moschella, 'Analyticity properties and thermal effects for general quantum field theory on de Sitter space-time," Commun. Math. Phys. **196** (1998) 535.

[8] J. Bros, H. Epstein and U. Moschella, "Towards a general theory of quantized fields on the anti-de Sitter space-time," Commun. Math. Phys. **231** (2002) 481.

[9] L. Randall and R. Sundrum, "A large mass hierarchy from a small extra dimension," Phys. Rev. Lett. **83** (1999) 3370 [hep-ph/9905221].

[10] M. Bertola, J. Bros, U. Moschella and R. Schaeffer, "A general construction of conformal field theories from scalar anti-de Sitter quantum field theories," Nucl. Phys. B **587** (2000) 619.

[11] K. H. Rehren, "Algebraic holography," Annales Henri Poincare **1** (2000) 607.